Stable Commutator Length

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Abstract

The notion of the commutator length was studied for a relatively long time under the guise of the *minimal genus problem.* Given fundamental group G of some topological space X and a loop γ in this space, which represents the conjugacy class of some group element g, the commutator length of g is a minimal genus of a surface S which admits a map to X which takes the boundary of S to γ.

The fact that the commutator length is not stable under taking powers of group elements gives rise to the theory of stable commutator length. Computing the value of stable commutator length is a very difficult problem, especially if one uses the original algebraic definition which turns out to be almost completely useless.

In this dissertation we give an overview of the theory of stable commutator length, following D. Calegari [4]. Together with the algebraic point of view, we discuss the geometric and functional analysis interpretations. Our ultimate goal is to systematize the introduction to the theory and to extend the proofs, where possible.

For instance, in section 1.4 we prove several results on the properties of *quasimorphisms* on groups, which we use in the study of stable commutator length. In section 1.6 we give a detailed proof of Bavard's Duality Theorem, which we then use in section 1.7 to show that stable commutator length vanishes in amenable groups.

As a broadening example, in section 1.8 we explain, following D. Kotschick [9], a general argument which can be used to prove that stable commutator length vanishes for certain groups defined as unions of subgroups which have many elementwise commuting conjugate embeddings.

In the second chapter of this dissertation we present a direct proof of the fact that stable commutator length takes only rational values on the elements of free groups of arbitrary rank.

Contents

Chapter 1

Stable Commutator Length

1.1 Basic definitions and properties

Definition 1.1.1. Let G be a group and $g \in [G, G]$ be a representative of the derived subgroup of G. Define the function $\mathrm{cl}_G : G \longrightarrow \mathbb{Z}$ such that $\mathrm{cl}_G(g)$ is the least integer k such that g is equal to a product of k commutators, i.e. $g = [a_1, b_1] \cdot \ldots \cdot [a_k, b_k]$. This is the *commutator length* of an element g. By convention we also define $\mathrm{cl}_G(g) = \infty$ in case $g \notin [G, G]$.

Definition 1.1.2. Let G be a group. For $g \in [G, G]$ we define the *stable commutator length*, denoted $\mathrm{scl}_G(g)$ to be the limit

$$\mathrm{scl}_G(g) = \lim_{n\to\infty} \frac{\mathrm{cl}_G(g^n)}{n}. \tag{1.1}$$

Suppose that $g_1, g_2 \in [G, G]$ with $\mathrm{cl}_G(g_1) = k$ and $\mathrm{cl}_G(g_2) = l$. Then $g_1 g_2$ can be expressed as a product of $k + l$ commutators and clearly $\mathrm{cl}_G(g_1 g_2) \leqslant k + l$. Hence, the commutator length is *subadditive*.

The very first question one might ask is whether the limit (1.1) exists. To answer this question we recall the well-known Fekete's lemma.

Lemma 1.1.3. [Fekete] Let $\{x_n\}_{n=1}^{\infty}$ be a subadditive sequence of real numbers with each $x_i \geqslant 0$. Then the sequence $\left\{\frac{x_n}{n}\right\}_{n=1}^{\infty}$ is bounded below and $\frac{x_n}{n}$ converges to $\inf_{n\in\mathbb{Z}_+} \left\{\frac{x_n}{n}\right\}$.

Proof. First of all, since $x_n \geqslant 0$ for each $n \in \mathbb{Z}_+$, it follows that $\frac{x_n}{n} \geqslant 0$, so the sequence $\left\{\frac{x_n}{n}\right\}_{n=1}^{\infty}$ is bounded below.

Denote $L = \inf_{n\in\mathbb{Z}_+} \left\{\frac{x_n}{n}\right\}$. We aim to show that $\frac{x_n}{n} \to L$ as $n \to \infty$. Let $\varepsilon > 0$ be arbitrary and let the index $k \in \mathbb{Z}_+$ be such that

$$\left|\frac{x_k}{k} - L\right| < \frac{\varepsilon}{2}.$$

Such k exists, since if not, then $L + \frac{\varepsilon}{2}$ would also be a lower bound, which contradicts the fact that L is the greatest lower bound by definition. Now let the index $m \in \mathbb{Z}_+$ be such that

$$\frac{x_r}{km} < \frac{\varepsilon}{2}$$

for all $r < k$. In order to do satisfy this condition, it is enough to pick m such that $\frac{M}{m} < \frac{\varepsilon}{2}$, where $M = \max_{r<k} \left\{ \frac{x_r}{k} \right\}$.

Now denote $N = km$ and let $n \geqslant N$ be arbitrary. We can write $n = pk + q$, where $p, q \in \mathbb{Z}$ such that $0 \leqslant q < k$. Since $n \geqslant N = km$, it follows that $p \geqslant m$, and we have

$$\frac{x_n}{n} \leqslant \frac{px_k}{pk+q} + \frac{x_q}{pk+q} \leqslant \frac{px_k}{pk} + \frac{x_q}{km} = \frac{x_k}{k} + \frac{x_q}{km} < \left(L + \frac{\varepsilon}{2}\right) + \frac{\varepsilon}{2},$$

since $\frac{x_k}{k} < L + \frac{\varepsilon}{2}$ and $x_{pk} \leqslant x_{(p-1)k} + x_k \leqslant \ ... \ \leqslant px_k$. As $\frac{x_n}{n} \geqslant L$, it follows that $\left|\frac{x_n}{n} - L\right| < \varepsilon$. Therefore we have shown that there exists $N \in \mathbb{Z}_+$ such that for all $n \geqslant N$ we have $\left|\frac{x_n}{n} - L\right| < \varepsilon$ since $n \geqslant N$ is arbitrary. Finally, since $\varepsilon > 0$ is also arbitrary, we have

$$\forall \varepsilon > 0 \ \exists N \in \mathbb{Z}_+ : \ \forall n \geqslant N \ \left|\frac{x_n}{n} - L\right| < \varepsilon,$$

which means that $\frac{x_n}{n} \to L$ as $n \to \infty$. □

Corollary 1.1.4. The limit

$$\mathrm{scl}_G(g) = \lim_{n\to\infty} \frac{\mathrm{cl}_G(g^n)}{n}$$

exists.

If for some element $g \notin [G, G]$ there exists an integer $n \in \mathbb{Z}_+$ such that $g^n \in [G, G]$, we define

$$\mathrm{scl}_G(g) = \frac{\mathrm{scl}_G(g^n)}{n}.$$

Now we shall discuss some basic properties of commutator length and stable commutator length. One of their main properties is that they are both monotone under homomorphisms:

Lemma 1.1.5. Let $\varphi : G \longrightarrow H$ be a group homomorphism. Then the following inequalities hold:

$$\begin{array}{rcl} \mathrm{cl}_G(g) & \geqslant & \mathrm{cl}_H(\varphi(g)), \\ \mathrm{scl}_G(g) & \geqslant & \mathrm{scl}_H(\varphi(g)) \end{array}$$

for all $g \in G$.

Proof. Recall that the image of a commutator under a homomorphism is a commutator:

$$\varphi([a,b]) = [\varphi(a), \varphi(b)]$$

for all $a, b \in G$. Now suppose that for an element $g \in G$ we have $\mathrm{cl}_G(g) = k$, i.e. $g = [a_1, b_1] \cdot ... \cdot [a_k, b_k]$. Then $\varphi(g) = [\varphi(a_1), \varphi(b_1)] \cdot ... \cdot [\varphi(a_k), \varphi(b_k)]$, so $\mathrm{cl}_H(\varphi(g)) \leqslant k$. Thus $\mathrm{cl}_G(g) \geqslant \mathrm{cl}_H(\varphi(g))$ and since

$$\frac{\mathrm{cl}_G(g^n)}{n} \geqslant \frac{\mathrm{cl}_H((\varphi(g))^n)}{n},$$

we also have $\mathrm{scl}_G(g) \geqslant \mathrm{scl}_H(\varphi(g))$. □

Corollary 1.1.6. Suppose $\varphi : G \longrightarrow H$ is a monomorphism with a left inverse. Then

$$\mathrm{scl}_G(g) = \mathrm{scl}_H(\varphi(g))$$

for all $g \in G$.

Proof. Suppose $\psi : H \longrightarrow G$ is a left inverse for φ, i.e. $\psi \circ \varphi = \mathrm{id}_G$. From the Lemma 1.1.5 it follows that for an element $g \in G$

$$\mathrm{scl}_G(g) \geqslant \mathrm{scl}_H(\varphi(g)).$$

Since $g = \psi(\varphi(g))$, we also have

$$\mathrm{scl}_G(g) = \mathrm{scl}_G(\psi(\varphi(g))) \leqslant \mathrm{scl}_H(\varphi(g)),$$

and hence

$$\mathrm{scl}_G(g) = \mathrm{scl}_H(\varphi(g)).$$

□

Another interesting property of stable commutator length is the existence of a countable subgroup which preserves it for a given element:

Lemma 1.1.7. Given a group G and an element $g \in G$. There exists a countable subgroup $H < G$ such that $g \in H$ and $scl_H(g) = scl_G(g)$.

Proof. For each $n \in \mathbb{Z}_+$ construct a subgroup H_n of G in a following way: suppose $\mathrm{cl}_G(g^n) = k$, i.e. $g^n = [a_1, b_1] \cdot ... \cdot [a_k, b_k]$. Let H_n be a subgroup of G generated by the elements $\{a_1, ..., a_k, b_1, ..., b_k\}$:

$$H_n = \langle a_1, ..., a_k, b_1, ..., b_k \rangle < G.$$

Clearly, $\mathrm{cl}_{H_n}(g^n) = \mathrm{cl}_G(g^n)$ and taking $H = \bigcup_n H_n$ finishes the proof. □

1.2 Surfaces, commutators and the fundamental group

In this section we discuss some of the properties of topological surfaces.

A *surface* is a two-dimensional topological manifold. That is, a surface is a topological space in which every point has an open neighbourhood homeomorphic to some open subset of the Euclidean plane, usually denoted by $\mathbb{R}^2$. It is also often assumed that a surface as a topological space is paracompact and Hausdorff; that is, every open cover has a locally finite refinement and any two distinct points have disjoint neighbourhoods.

A *surface with boundary* is a Hausdorff topological space in which every point has an open neighbourhood homeomorphic to some open subset of either the Euclidean plane or the upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y \geqslant 0\}$. Points of a surface S, whose neighbourhood is homeomorphic to the upper half-plane form the *boundary* of a surface which is denoted by ∂S. Those points of S which are not contained in ∂S form the *interior* of a surface which is denoted by $\text{int}(S)$. A surface without boundary, which is compact is called a *closed surface*. In the following discussion the word *surface* will refer to surfaces without boundary. We also say that a topological surface is of *finite type* if it is homeomorphic to a closed surface with a finite number of points removed.

Recall also that the surface S is *orientable*, if any loop going around one way on the surface can never be continuously deformed without overlapping itself to a loop going around the opposite way. If S is not orientable, then it is called *non-orientable*.

In 1925 T. Radó proved (see [15]) that every topological surface S can be triangulated, that is, there exists a finite family of closed subsets $\mathcal{T} = \{T_1, ..., T_n\}$ and a family of homeomorphisms $\varphi_i : \Delta_i^2 \longrightarrow T_i$, $i = 1, ..., n$, where each Δ_i^2 is a triangle on a plane $\mathbb{R}^2$. The elements of $\mathcal{T}$ are called the *faces* of a triangulation. Images of the vertices and edges of a triangle Δ_i^2 under homeomorphism φ_i are called vertices and edges of a triangulation respectively. It is also required that any two distinct elements T_i and T_j of $\mathcal{T}$ are either disjoint, have one single edge in their intersection, or have a single vertex in their intersection.

Given a surface S with a triangulation $\mathcal{T}$ containing F faces, V vertices and E edges. Then the *Euler characteristic* (or *Euler-Poincaré characteristic*) is defined in the following way:

$$\chi(S) = V - E + F.$$

For example, Euler characteristic of a sphere S^2 is 2, Euler characteristic of a torus T is 0 and $\chi(P) = 1$, where P is a projective plane. It is a well-known result that every closed surface can be obtained as a connected sum of some finite number of copies of S^2, T and P. The connected sum is denoted by $\#$ and has the following properties: it is associative, commutative, has the sphere S^2 as an identity and also satisfies the following condition:

$$T\#P = P\#P\#P.$$

Euler characteristic of a connected sum of surfaces S_1 and S_2 is given by

$$\chi(S_1\#S_2) = \chi(S_1) + \chi(S_2) - 2.$$

A surface homeomorphic to a connected sum of g copies of tori T or projective plane P is said to be of *genus* g. A sphere S^2 is said to be of *genus* 0. There is a relation between the genus g of a surface S and the Euler characteristic $\chi(S)$:

$$g = \begin{cases} \frac{1}{2}(2 - \chi(S)) & \text{if } S \text{ is orientable,} \\ 2 - \chi(S) & \text{if } S \text{ is non-orientable.} \end{cases}$$

Suppose S is an oriented surface of finite type of genus g with p punctures, where $p > 0$. Then the van Kampen's theorem gives us the method for determining the fundamental group of S based on the division of the surface S into parts for which the fundamental group is known. In our case it can be shown that $\pi_1(S)$ is a free group of rank $2g + p - 1$. This follows from the fact that S retracts onto a rose with $2g + p - 1$ leaves, so the fundamental group of S is a free product of $2g + p - 1$ copies of the fundamental group of the circle S^1:

$$\pi_1(S) = \underbrace{\pi_1(S^1) * ... * \pi_1(S^1)}_{2g+p-1} = \underbrace{\mathbb{Z} * ... * \mathbb{Z}}_{2g+p-1} = F_{2g+p-1}.$$

Recall that a subspace Y of space X is called a *retract* of X if there is a continuous map $r : Y \longrightarrow X$ such that $r(Y) = Y$. For example, a sphere with $p > 0$ punctures retracts onto a rose with $p - 1$ leaves, and thus its fundamental group is a free group of rank $p - 1$ (Figure 1.1).

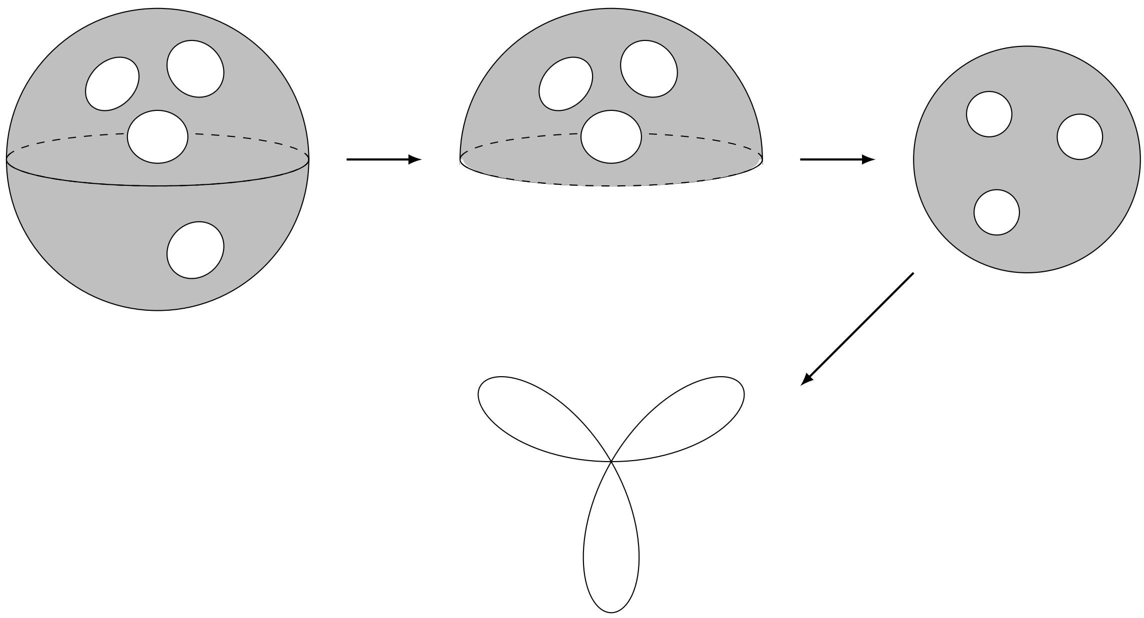

Figure 1.1: A sphere with p punctures retracts onto a rose with $p-1$ leaves.

Suppose that S is a closed surface of genus g. Then it can be obtained from a $4g$-gon by "gluing" the edges in pairs and the van Kampen's theorem gives us the presentation of the fundamental group $\pi_1(S)$:

$$\pi_1(S) = \langle a_1, b_1, ..., a_g, b_g \mid [a_1, b_1] \cdot ... \cdot [a_g, b_g] \rangle. \tag{1.2}$$

For example, consider the fundamental group of the torus T, which is represented as a square with identification. Let V be a smaller open disk and U be the whole figure without a closed disk which is smaller than V, so that their overlap W is an open annulus (Figure 1.2).

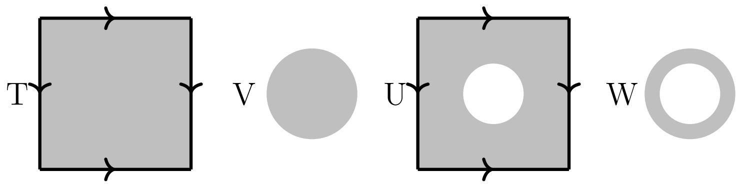

Figure 1.2: Computation of the fundamental group of the torus T.

Then, clearly, U is homotopy equivalent to its boundary. Take $S_U = \{a, b\}$ corresponding to the labelled loops. The fundamental group $\pi_1(V)$ is trivial, since V is contractible and W is homotopy equivalent to a circle S^1, hence $\pi_1(W)$ has one generator, a loop γ that runs around the annulus. Including this loop into V makes it null-homotopic, represented by an empty word, and including γ into U makes it homotopic to the path running round the boundary of a square, which can be represented in terms of the "coordinates" S_U as a word $a^{-1}b^{-1}ab$. Thus, by the van Kampen's theorem,

$$\pi_1(T) = \langle a, b \mid a^{-1}b^{-1}ab \rangle = \langle a, b \mid [a, b] \rangle.$$

In general, the presentation of the fundamental group of a closed surface of genus g can be obtained by applying the same method which gives the result (1.2).

Any closed surface of genus g can be obtained from a surface S of genus g with one boundary component by gluing on a disk. We already know that the fundamental group of S is free of rank $2g$, so we can say that $\pi_1(S)$ has generators $a_1, b_1, ..., a_g, b_g$ and the conjugacy class of the element $[a_1, b_1] \cdot ... \cdot [a_g, b_g]$ is represented by ∂S.

Let X be a topological space and let $\gamma \subset X$ be a loop. Suppose $a \in \pi_1(X)$ is a conjugacy class represented by γ such that a has a representative in the derived subgroup $(\pi_1(X))' = [\pi_1(X), \pi_1(X)]$, i.e. we can write

$$a = [\alpha_1, \beta_1] \cdot ... \cdot [\alpha_g, \beta_g].$$

Let S be a surface of genus g with one boundary component. It can be obtained from a $(4g+1)$-gon P by identifying the edges in pairs. We can choose loops in a space X which represent the elements a, α_i and β_i and define a map $f : \partial P \longrightarrow X$ sending the edges a_i and b_i to α_i and β_i respectively and also sending one remaining edge of P to a. Then, by construction, $f = \widetilde{f} \circ g$, where the quotient map $g : \partial P \longrightarrow S$ is induced by identifying all but one of the edges of P in pairs. Moreover, $f(\partial P)$ is null-homotopic in X, so f can be extended to a map $S \longrightarrow X$ which sends ∂S to a. This means that loops corresponding to elements of the derived subgroup of the fundamental group of X bound maps of oriented surfaces into X.

1.3 Geometric interpretation of stable commutator length

In many cases the original algebraic definitions of the commutator length and stable commutator length are almost completely useless. It is to be noted that powers and products of commutators are not well-behaved in terms of the commutator length. For instance, we recall the well-known example presented by Culler [5]:

$$[a,b]^3 = [aba^{-1}, b^{-1}aba^{-2}][b^{-1}ab, b^2].$$

This identity holds for any elements a and b in any group.

Let G be a group. We can construct a topological space X such that $\pi_1(X) = G$ (see, for example, [8]). Suppose $a \in G$ is a conjugacy class, then a corresponds to a free homotopy class of loop γ in X. From the discussion in the previous section it follows that $\mathrm{cl}_G(a)$ is the least genus of a surface S with one boundary component, which maps to X in a way that the boundary represents the free homotopy class of γ. Then we can obtain $\mathrm{scl}_G(a)$ by estimating the genus of surfaces whose boundary wraps multiple times around γ.

Suppose S is a compact oriented surface. Define

$$-\chi^-(S) = \sum_i \max(-\chi(S_i), 0),$$

where S_i are the components of S. Given a group G, a topological space X with $\pi_1(X) = G$ and a loop $\gamma : S^1 \longrightarrow X$, we call a map $f : S \longrightarrow X$ *admissible* if there is a commutative diagram

$$\begin{array}{ccc} \partial S & \xrightarrow{\ i\ } & S \\ {\scriptstyle \partial f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ S^1 & \xrightarrow[\ \gamma\ ]{} & X \end{array}$$

where $i : \partial S \longrightarrow S$ is the inclusion map. Since S is oriented, ∂S has an inherited orientation and we can define the fundamental class $[\partial S] \in H_1(\partial S)$. Similarly we can define a fundamental class $[S^1]$ in $H_1(S^1)$. Denote the oriented components of ∂S by ∂_i and define $n(S)$ as the sum of the degrees of the maps $\partial f_i : \partial_i \longrightarrow S^1$, by the following identity:

$$\partial f_*[\partial S] = n(S)[S^1].$$

Informally speaking, $n(S)$ is the degree with which ∂S wraps around the loop γ. If $n(S) \neq 0$, then S is said to *rationally bound* γ. By choosing the orientation of S properly, we can always ensure that $n(S) \geqslant 0$.

Now we are ready to give the geometric definition of stable commutator length.

Proposition 1.3.1. Given a group G and a topological space X with $\pi_1(X) = G$. Let $\gamma : S^1 \longrightarrow X$ be a loop in X representing the conjugacy class of an element $a \in G$. Then

$$\mathrm{scl}_G(a) = \inf_S \frac{-\chi^-(S)}{2n(S)} \tag{1.3}$$

where the infimum is taken over all admissible maps $f : S \longrightarrow X$ as defined above.

Proof. Suppose that S is a surface of genus g with one boundary component. Note that $\mathrm{cl}_G(a^n) \leqslant g$ if and only if there is an admissible map $f : S \longrightarrow X$ and S satisfies $n(S) = n$ and $-\chi^-(S) = 2g - 1$. Thus,

$$\mathrm{scl}_G(a) = \lim_{n\to\infty} \frac{\mathrm{cl}_G(a^n)}{n} \geqslant \inf_S \frac{-\chi^-(S)}{2n(S)}.$$

Suppose now $f : S \longrightarrow X$ is admissible. If S has multiple components $\{S_i\}_{i=1}^k$, then clearly there is at least one component S_i such that

$$\frac{-\chi^-(S_i)}{2n(S_i)} \leqslant \frac{-\chi^-(S)}{2n(S)}.$$

This means we can assume that S is connected without loss of generality. Note also that both $-\chi^-(S)$ and $n(S)$ are multiplicative under coverings and passing to a cover multiplies both of them by the same factor. Hence, we can replace S with a finite cover S' without changing the ratio:

$$\frac{-\chi^-(S)}{2n(S)} = \frac{-\chi^-(S')}{2n(S')}.$$

We can also assume that S has $p > 1$ boundary components.

If $S' \longrightarrow S$ is a finite cover of S of degree $N > 0$ with p boundary components, then $-\chi^-(S') = -N\chi^-(S)$ and $n(S') = Nn(S)$. Two different boundary components can be connected together with 1-handle, whilst $\partial f'$ being extended by a trivial map to a basepoint of the circle S^1. If a 1-handle connects two different boundary components, then those two components have been absorbed into one along the boundary of the 1-handle. In other words, such an operation increases the genus by one and reduces the number of boundary components by one, so it increases the value of $-\chi^-$ by one. Applying this operation multiple times, we get a surface S'' with one boundary component satisfying $-\chi^-(S'') = -\chi^-(S') + p - 1$ and $n(S'') = n(S')$. This leads to the following equality:

$$\frac{-\chi^-(S'')}{2n(S'')} = \frac{p - 1 - N\chi^-(S)}{2Nn(S)}.$$

Taking N sufficiently large allows the right-hand side of this equality to be arbitrary close to

$$\inf_S \frac{-\chi^-(S)}{2n(S)}.$$

On the other hand, since S'' has exactly one boundary component and since we may choose the genus of S'' to be sufficiently large, we have

$$\mathrm{cl}_G(a^{n(S'')}) \leqslant \frac{-\chi^-(S'')}{2} + 1,$$

and hence, combining with the first part of the proof, we get

$$\mathrm{scl}_G(a) = \inf_S \frac{-\chi^-(S)}{2n(S)}.$$

□

Definition 1.3.2. A surface S with an admissible map $f : S \longrightarrow X$, which realizes the infimum (1.3) is called *extremal*.

1.4 Quasimorphisms

Together with the algebraic and geometric definitions, there is also a functional analysis definition of stable commutator length, which is given in terms of the *quasimorphisms* on groups. This way leads us to the *Bavard duality*, which will be discussed later.

Definition 1.4.1. Let G be a group. A *quasimorphism* is a function $\varphi : G \longrightarrow \mathbb{R}$ for which there is a least constant $D(\varphi) \geqslant 0$ such that

$$|\varphi(g_1g_2) - \varphi(g_1) - \varphi(g_2)| \leqslant D(\varphi)$$

for all $g_1, g_2 \in G$. The constant $D(\varphi)$ is called the *defect* of φ.

Clearly, for a quasimorphism φ, $D(\varphi) = 0$ if and only if φ is a homomorphism. Note also that any bounded function is a quasimorphism.

Lemma 1.4.2. Let G be a group and let S be its generating set, which can be infinite. Let $g \in G$ and let ω to be the word in generators representing g. Denote by ω_i the i-th letter of the word ω and let also $|\omega|$ denote the length of the word ω. Then there is an inequality

$$\left|\varphi(\omega) - \sum_{i=1}^{|\omega|} \varphi(\omega_i)\right| \leqslant (|\omega| - 1)D(\varphi),$$

where $\varphi : G \longrightarrow \mathbb{R}$ is a quasimorphism with the defect $D(\varphi)$.

Proof. We proceed by induction on the length of the word $|\omega|$. The case $|\omega| = 1$ is trivial. Suppose that $|\omega| = 2$, i.e. $\omega = \omega_1\omega_2$. Then the desired inequality is precisely the definition of a quasimorphism:

$$|\varphi(\omega) - \varphi(\omega_1) - \varphi(\omega_2)| \leqslant D(\varphi).$$

Now suppose that the inequality holds for all the words ω with $|\omega| \leqslant n$ and consider some word $\widetilde{\omega}$ with $|\widetilde{\omega}| = n + 1$. By the definition of a quasimorphism we have:

$$|\varphi(\widetilde{\omega}) - \varphi(\widetilde{\omega}') - \varphi(\widetilde{\omega}_{n+1})| \leqslant D(\varphi),$$

where $|\widetilde{\omega}'| = n$. This inequality is equivalent to

$$\varphi(\widetilde{\omega}') - D(\varphi) \leqslant \varphi(\widetilde{\omega}) - \varphi(\widetilde{\omega}_{n+1}) \leqslant \varphi(\widetilde{\omega}') + D(\varphi).$$

By the inductive hypothesis,

$$\sum_{i=1}^{n} \varphi(\widetilde{\omega}'_i) - (n-1)D(\varphi) \leqslant \varphi(\widetilde{\omega}') \leqslant \sum_{i=1}^{n} \varphi(\widetilde{\omega}'_i) + (n-1)D(\varphi),$$

so overall we have

$$\left|\varphi(\widetilde{\omega}) - \sum_{i=1}^{n+1} \varphi(\widetilde{\omega}_i)\right| \leqslant nD(\varphi) = ((n+1)-1)D(\varphi),$$

and the proof follows. □

For a fixed group G, the set of all quasimorphisms on G form a real vector space denoted by $\widehat{Q}(G)$. Some of the quasimorphisms have properties, which can be extremely useful in our further study.

Definition 1.4.3. A quasimorphism $\varphi : G \longrightarrow \mathbb{R}$ is said to be *homogeneous* if it satisfies the following property:

$$\varphi(g^n) = n\varphi(g)$$

for all $g \in G$ and $n \in \mathbb{Z}$. Homogeneous quasimorphisms form a real vector space denoted by $Q(G)$.

Lemma 1.4.4. Let φ be a quasimorphism on a group G. For each element $g \in G$ define

$$\widehat{\varphi}(g) = \lim_{n\to\infty} \frac{\varphi(g^n)}{n}.$$

The limit exists and $\widehat{\varphi}$ is a homogeneous quasimorphism on G. There is also an estimate $|\widehat{\varphi}(g) - \varphi(g)| \leqslant D(\varphi)$.

Proof. Let i be a positive integer. From the definition of a quasimorphism it follows that

$$\left|\varphi(g^{2^i}) - 2\varphi(g^{2^{i-1}})\right| \leqslant D(\varphi).$$

We claim that for any $j < i$,

$$\left|\varphi(g^{2^i}) \cdot 2^{j-i} - \varphi(g^{2^j})\right| \leqslant D(\varphi).$$

To prove this claim, we first prove by induction on j the following inequality:

$$\left|\varphi(g^{2^i}) - 2^{i-j}\varphi(g^{2^j})\right| \leqslant (2^{i-j} - 1)D(\varphi).$$

The case $j = i - 1$ follows directly from the definition of a quasimorphism. Now suppose that the inequality holds for all j such that $k \leqslant j < i$. Then we have:

$$\left|\varphi(g^{2^i}) - 2^{i-k+1}\varphi(g^{2^{k-1}})\right| = \left|\varphi(g^{2^i}) - 2^{i-k}\varphi(g^{2^k}) + 2^{i-k}\varphi(g^{2^k}) - 2^{i-k+1}\varphi(g^{2^{k-1}})\right| \leqslant$$
$$\leqslant (2^{i-k} - 1)D(\varphi) + 2^{i-k}D(\varphi) = (2^{i-k+1} - 1)D(\varphi),$$

so the inequality holds for all $j < i$. Division by 2^{i-j} proves the claim:

$$\left|\varphi(g^{2^i}) \cdot 2^{j-i} - \varphi(g^{2^j})\right| \leqslant \frac{2^{i-j} - 1}{2^{i-j}} \cdot D(\varphi) \leqslant D(\varphi).$$

Hence, the sequence $\left\{\varphi(g^{2^i}) \cdot 2^{-i}\right\}_{i=1}^{\infty}$ is a Cauchy sequence. Define

$$\widehat{\varphi}(g) = \lim_{i\to\infty} \left(\varphi(g^{2^i}) \cdot 2^{-i}\right),$$

and note that $|\widehat{\varphi}(g) - \varphi(g)| \leqslant D(\varphi)$ for all $g \in G$. Hence, $\widehat{\varphi} - \varphi$ is a bounded function and, moreover, $\widehat{\varphi}$ is a quasimorphism. Applying the definition of $\widehat{\varphi}$ and Lemma 1.4.2, we get

$$\left|\widehat{\varphi}(g^j) - j\widehat{\varphi}(g)\right| = \lim_{i\to\infty} \frac{\varphi(g^{j2^i}) - j\varphi(g^{2^i})}{2^i} \leqslant \lim_{i\to\infty} \frac{(j-1)D(\varphi)}{2^i} = 0,$$

so $\widehat{\varphi}$ is indeed homogeneous. □

The quasimorphism $\widehat{\varphi}$ from the previous lemma is called the *homogenization* of φ.

Lemma 1.4.5. Let G be a group and let $\varphi : G \longrightarrow \mathbb{R}$ be a homogeneous quasimorphism. If $g_1, g_2 \in G$ commute, then

$$\varphi(g_1 g_2) = \varphi(g_1) + \varphi(g_2).$$

Proof. Suppose $g_1, g_2 \in G$ commute. Since φ is homogeneous, we have

$$|\varphi(g_1 g_2) - \varphi(g_1) - \varphi(g_2)| = \lim_{n\to\infty} \frac{1}{n} |\varphi((g_1 g_2)^n) + \varphi(g_1^{-n}) + \varphi(g_2^{-n})| \leqslant$$
$$\leqslant \lim_{n\to\infty} \frac{1}{n} \left(|\varphi((g_1 g_2)^n g_1^{-n} g_2^{-n})| + 2D(\varphi)\right) =$$
$$= \lim_{n\to\infty} |\varphi((g_1 g_2)^n g_1^{-n} g_2^{-n})| = 0,$$

and the proof follows. □

Definition 1.4.6. A quasimorphism $\varphi : G \longrightarrow \mathbb{R}$ is said to be *antisymmetric*, if for all $g \in G$,

$$\varphi(g^{-1}) = -\varphi(g).$$

Any quasimorphism φ can be *antisymmetrized*:

$$\varphi'(g) = \frac{\varphi(g) - \varphi(g^{-1})}{2}.$$

Lemma 1.4.7. The antisymmetrization φ' of a quasimorphism φ satisfies

$$D(\varphi') \leqslant D(\varphi).$$

Proof. We estimate

$$D(\varphi') = \sup_{g_1,g_2\in G} |\varphi(g_1g_2) - \varphi(g_1) - \varphi(g_2)| =$$
$$= \sup_{g_1,g_2\in G} \frac{1}{2} \left|\varphi(g_1g_2) - \varphi(g_1) - \varphi(g_2) - \varphi(g_2^{-1}g_1^{-1}) + \varphi(g_1^{-1}) + \varphi(g_2^{-1})\right| \leqslant D(\varphi).$$

□

1.4.1 Commutator estimates

Lemma 1.4.8. Let φ be an antisymmetric quasimorphism on a group G and let a_i and b_i be the elements of G for $1 \leqslant i \leqslant n$. Then

$$\left|\varphi\left(\prod_{i=1}^{n}[a_i, b_i]\right)\right| \leqslant (4n-1)D(\varphi).$$

Proof. We prove the lemma by induction on n. If $n = 1$, then, by the definition of a quasimorphism, we have

$$\begin{array}{rcccl} -D(\varphi) & \leqslant & \varphi((a^{-1}b^{-1}a)b) - \varphi(a^{-1}b^{-1}a) - \varphi(b) & \leqslant & D(\varphi), \\ -D(\varphi) & \leqslant & \varphi((a^{-1}b^{-1})a) - \varphi(a^{-1}b^{-1}) - \varphi(a) & \leqslant & D(\varphi), \\ -D(\varphi) & \leqslant & \varphi(a^{-1}b^{-1}) - \varphi(a^{-1}) - \varphi(b^{-1}) & \leqslant & D(\varphi). \end{array}$$

Adding up these three inequalities and keeping in mind the fact that φ is antisymmetric, we get

$$|\varphi([a, b])| \leqslant 3D(\varphi),$$

as required.

Now suppose the inequality holds for some $n \geqslant 1$, i.e.

$$\left|\varphi\left(\prod_{i=1}^{n}[a_i, b_i]\right)\right| \leqslant (4n-1)D(\varphi).$$

Let g be a product of $n+1$ commutators:

$$g = [a_1, b_1] \cdot \ldots \cdot [a_{n+1}, b_{n+1}].$$

Then, from the definition of a quasimorphism it follows that

$$\begin{aligned} |\varphi(g)| \leqslant D(\varphi) + |\varphi([a_1, b_1] \cdot \ldots \cdot [a_n, b_n]) + \varphi([a_{n+1}, b_{n+1}])| \leqslant \\ \leqslant D(\varphi) + (4n-1)D(\varphi) + 3D(\varphi) = (4(n+1)-1)D(\varphi), \end{aligned}$$

finishing the proof. □

Note that any homogeneous quasimorphism is antisymmetric and also if φ is a homogeneous quasimorphism on a group G, then

$$\left|\varphi(g_1^{-1}g_2g_1) - \varphi(g_2)\right| = \frac{1}{n}\left|\varphi(g_1^{-1}g_2^n g_1) - \varphi(g^n)\right| \leqslant \frac{D(\varphi)}{n}$$

for all $g_1, g_2 \in G$. It follows that *homogeneous quasimorphisms are class functions*, i.e. they are constant on conjugacy classes. As a consequence, for any $a, b \in G$,

$$|\varphi([a, b])| \leqslant D(\varphi).$$

Indeed, since $\varphi(aba^{-1}) = \varphi(b)$, then

$$|\varphi([a, b])| = |\varphi(aba^{-1}b^{-1}) - \varphi(aba^{-1}) - \varphi(b^{-1})| \leqslant D(\varphi).$$

Lemma 1.4.9. Let G be a group and let $\varphi : G \longrightarrow \mathbb{R}$ be a homogeneous quasimorphism. Then

$$\varphi\left(\prod_{i=1}^{n}[a_i, b_i]\right) \leqslant 2nD(\varphi),$$

where $a_i, b_i \in G$, $1 \leqslant i \leqslant n$.

Proof. We have shown that

$$\varphi([a, b]) \leqslant D(\varphi) \leqslant 2D(\varphi).$$

Suppose that the inequality holds for all n such that $n \leqslant m$, then

$$\left|\varphi\left(\prod_{i=1}^{m+1}[a_i, b_i]\right) - \varphi\left(\prod_{i=1}^{m}[a_i, b_i]\right) - \varphi([a_{m+1}, b_{m+1}])\right| \leqslant D(\varphi).$$

It follows that

$$\left|\varphi\left(\prod_{i=1}^{m+1}[a_i, b_i]\right)\right| \leqslant 2mD(\varphi) + D(\varphi) + D(\varphi) = 2(m+1)D(\varphi),$$

so the inequality holds for $n = m + 1$, which proves the lemma. □

Lemma 1.4.10. [Bavard] Let G be a group and let $\varphi : G \longrightarrow \mathbb{R}$ be a homogeneous quasimorphism. Then there is an equality

$$\sup_{a,b \in G} |\varphi([a, b])| = D(\varphi).$$

Proof. Consider an element $g = a^{2n}b^{2n}(ab)^{-2n}$. We claim that g can be expressed as a product of n commutators. In case $n = 1$ we have:

$$a^2b^2(ab)^{-2} = a^2b^2b^{-1}a^{-1}b^{-1}a^{-1} = a^2ba^{-1}b^{-1}a^{-1} = a[a, b]a^{-1} = [a, aba^{-1}].$$

Note that also

$$a^{2n}b^{2n}(ab)^{-2n} = a^{2n}b^{2n}(b^{-1}a^{-1})^{2n} = \\ = a^{2n}b^{2n-1}(a^{-1}b^{-1})^{2n-1}a^{-1} = a(a^{2n-1}b^{2n-1}(ba)^{-(2n-1)})a^{-1},$$

so we only need to show that $a^{2n-1}b^{2n-1}(ba)^{-(2n-1)}$ can be written as a product of n commutators. Assume that we have proved this for all $n \leqslant m$, then

$$[a^{-2m+1}b^{-2m}a^{-2}, ab^{-1}a^{2m-1}] = a^{-2m+1}b^{-2m}a^{-1}b^{-1}a^{2m+1}b^{2m+1}a^{-1} = \\ = a(a^{-2m}b^{-2m}a^{-1}b^{-1}a^{2m+1}b^{2m+1})a^{-1}.$$

By the inductive hypothesis, after interchanging a and b for a^{-1} and b^{-1} respectively, $a^{-2m}b^{-2m}$ can be expressed as a product of m commutators and $(a^{-1}b^{-1})^{2m}$, so $(a^{-1}b^{-1})^{2m+1}a^{2m+1}b^{2m+1}$ can be expressed as a product of $m+1$ commutators, hence, the inductive step is complete and the claim holds for $n = m+1$.

Now choose some $a, b \in G$ such that

$$|\varphi(ab) - \varphi(a) - \varphi(b)| \geqslant D(\varphi) - \varepsilon,$$

where $\varepsilon > 0$ is arbitrary. Since φ is homogeneous, then for any n we have

$$|\varphi((ab)^{2n}) - \varphi(a^{2n}) - \varphi(b^{2n})| \geqslant 2n(D(\varphi) - \varepsilon).$$

We have just shown that $(ab)^{2n}$ can be written as a product of n commutators $[a_i, b_i]$ and $a^{2n}b^{2n}$, where each a_i and b_i depends on a and b:

$$(ab)^{2n} = [a_1, b_1] \cdot ... \cdot [a_n, b_n] \cdot a^{2n}b^{2n}.$$

Then, by Lemma 1.4.2 we have:

$$\left| \varphi((ab)^{2n}) - \varphi(a^{2n}) - \varphi(b^{2n}) - \sum_{i=1}^{n} \varphi([a_i, b_i]) \right| \leqslant (n+1)D(\varphi),$$

so, by the triangle inequality,

$$\left| \sum_{i=1}^{n} \varphi([a_i, b_i]) \right| \geqslant (n-1)D(\varphi) - 2n\varepsilon.$$

Since $|\varphi([a_i, b_i])| \leqslant D(\varphi)$, taking sufficiently large n and ε much smaller than $\frac{1}{n}$, we can ensure that some $\varphi([a_i, b_i])$ is as close to $D(\varphi)$ as we wish. □

Example 1.4.11. Let $F = F\langle a, b \rangle$ be a free group of rank two. Define a function $t : \mathbb{Z} \longrightarrow \mathbb{Z}$ by

$$t(m) = \begin{cases} 0 & \text{if } m \equiv 0 \pmod 3; \\ 1 & \text{if } m \equiv 1 \pmod 3; \\ -1 & \text{if } m \equiv -1 \pmod 3). \end{cases}$$

It is easy to see that for any two integers m and n, the following inequality holds:

$$|t(m+n) - t(m) - t(n)| \leqslant 3.$$

Define now a function $\varphi_a : F \longrightarrow \mathbb{Z}$ by

$$\varphi_a(\omega) = \sum_{i=1}^{n} t(p_i),$$

if

$$\omega = a^{p_1} b^{q_1} ... a^{p_n} b^{q_n},$$

where $p_i, q_i \in \{-1, 1\}$ with the following exception: $p_1, q_n \in \{-1, 0, 1\}$. It is easy to see that $\varphi_a(\omega^{-1}) = -\varphi_a(\omega)$. Furthermore, suppose that $u, v \in F$ and $u = u_1...u_r$, $v = v_1...v_s$. Then $uv = u_1...u_{r-1} w v_2...v_s$, where $w = u_r v_1$. Then

$$\varphi_a(uv) = \sum_{i=1}^{r-1} \varphi_a(u_i) + \varphi_a(w) + \sum_{i=2}^{s} \varphi_a(v_i).$$

It follows that

$$|\varphi_a(uv) - \varphi_a(u) - \varphi_a(v)| = |\varphi_a(w) - \varphi_a(u_r) - \varphi_a(v_1)| \leqslant 3,$$

so φ_a is a quasimorphism. Moreover, it is easy to see that φ_a is homogeneous.

Consider a cyclically reduced word $\omega \in [F, F]$. Suppose that ω can be expressed as a product of m commutators. Then, by Lemma 1.4.9 we have

$$|\varphi_a(\omega)| \leqslant 2mD(\varphi_a) \leqslant 6m,$$

i.e. $m \geqslant \frac{|\varphi_a(\omega)|}{6}$, and since φ_a is homogeneous, we finally have

$$\mathrm{cl}_G(\omega^n) \geqslant \frac{n|\varphi_a(\omega)|}{6}.$$

1.4.2 Counting quasimorphisms

Now we shall discuss the *counting quasimorphisms* as one of the fundamental examples of quasimorphisms.

Definition 1.4.12. Let $F = F\langle S \rangle$ be a free group on a symmetric generating set S and let ω be a reduced word in S. Define the *big counting function* $C_\omega(g)$ to be the number of (possibly overlapping) occurrences of ω as a subword in the reduced representative of g. Define also the *small counting function* $c_\omega(g)$ as the maximum number of disjoint copies of ω as a subword in the reduced representative of g. The *big counting quasimorphism* H_ω is then defined by

$$H_\omega(g) = C_\omega(g) - C_{\omega^{-1}}(g).$$

Similarly, the *small counting quasimorphism* h_ω is defined by

$$h_\omega(g) = c_\omega(g) - c_{\omega^{-1}}(g).$$

Big counting quasimorphisms were introduced by R. Brooks [2], so C_ω and H_ω are often called *Brooks functions* and *Brooks quasimorphisms* respectively. Small counting functions were introduced by D. Epstein and K. Fujiwara [6]. Big counting quasimorphisms are usually easier to compute and easier to work with, whilst small counting quasimorphisms can be a more "powerful" tool, since they have uniformly small defects. Note that if no proper suffix (i.e. suffix which is not the whole word) of ω is equal to a proper prefix of ω, then all the occurrences of ω in the reduced representative of any $g \in F$ are disjoint, hence $H_\omega = h_\omega$ in this case.

To calculate the defects of H_ω and h_ω explicitly, we should first prove some preliminary results.

Lemma 1.4.13. Let F be a free group from Definition 1.4.12. Let $g \in F$ be reduced. Then the copies of ω in g are disjoint from the copies of ω^{-1}.

Proof. Suppose that some prefix of ω equals to some suffix of ω^{-1}. In this case $\omega = \omega_1\omega_2$, $\omega^{-1} = \omega_2^{-1}\omega_1^{-1}$ and $\omega_1 = \omega_1^{-1}$, which is impossible. □

Let $g \in F$ be such that $g = g_1 g_2$ as a reduced word, i.e. there is no cancellation of some suffix of g_1 with some prefix of g_2. An occurrence of ω is said to *intersect the "border"* of g if it overlaps both some nonempty suffix of g_1 and some nonempty prefix of g_2. Note that by Lemma 1.4.13 only one of ω and ω^{-1} can intersect the "border" of g.

Definition 1.4.14. Let $g \in F$ be a reduced expression such that $g = g_1 g_2$ and let ω be a reduced word. Define the *sign* of the expression, denoted s as

$$s = \begin{cases} 1 & \text{if } \omega \text{ intersects the ``border''}; \\ -1 & \text{if } \omega^{-1} \text{ intersects the ``border''}; \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 1.4.15. Let $g = g_1 g_2$ be a reduced expression with sign s. Then the value $h_\omega(g) - h_\omega(g_1) - h_\omega(g_2)$ is equal to either 0 or s, and

$$0 \leqslant s(H_\omega(g) - H_\omega(g_1) - H_\omega(g_2)) \leqslant |\omega| - 1.$$

Proof. Let X_1 be the maximal set of disjoint copies of ω in g_1 and X_2 be the maximal set of disjoint copies of ω in g_2. Then all the copies from $X_1 \cup X_2$ are contained in $g_1 g_2$, so we have

$$c_\omega(g) - c_\omega(g_1) - c_\omega(g_2) \geqslant 0.$$

Conversely, let X be the maximal set of disjoint copies of ω in $g = g_1 g_2$. Then either $X = X_1 \cup X_2$ or X contains one copy of ω which intersects the "border". Thus, if $s = 1$, then

$$c_\omega(g) - c_\omega(g_1) - c_\omega(g_2) \leqslant 1,$$

and

$$c_\omega(g) - c_\omega(g_1) - c_\omega(g_2) \leqslant 0$$

otherwise, and this proves the first inequality. The second inequality follows from the fact that at most $|\omega| - 1$ copies of ω or ω^{-1} can intersect the "border" of g. □

Note that H_ω is antisymmetric, since $C_\omega(g^{-1}) = C_{\omega^{-1}}(g)$. It follows that

$$D(H_\omega) \leqslant 3(|\omega| - 1).$$

Indeed, suppose $\widetilde{g}_1 = g_1 u$, $\widetilde{g}_2 = u^{-1} g_2$ and $g = \widetilde{g}_1 \widetilde{g}_2 = g_1 g_2$. Then by the previous lemma,

$$\begin{array}{rcccl} 0 & \leqslant & s_1(H_\omega(g) - H_\omega(\widetilde{g}_1) - H_\omega(\widetilde{g}_2)) & \leqslant & |\omega| - 1, \\ 0 & \leqslant & s_2(H_\omega(\widetilde{g}_1) - H_\omega(g_1) - H_\omega(u)) & \leqslant & |\omega| - 1, \\ 0 & \leqslant & s_3(H_\omega(\widetilde{g}_2) - H_\omega(u^{-1}) - H_\omega(g_2)) & \leqslant & |\omega| - 1. \end{array}$$

It only remains to add up these three inequalities to get the result $D(H_\omega) \leqslant 3(|\omega| - 1)$. Similarly, for small counting quasimorphisms one can obtain $D(h_\omega) \leqslant 3$, and with more effort it is possible to find an upper bound which is even smaller.

Proposition 1.4.16. Let ω be a reduced word. Then

(i) $D(h_\omega) = 0$ if and only if $|\omega| = 1$;

(ii) $D(h_\omega) = 2$ if and only if ω is of the form $\omega = \omega_1\omega_2\omega_1^{-1}$, $\omega = \omega_1\omega_2\omega_1^{-1}\omega_3$ or $\omega = \omega_1\omega_2\omega_3\omega_2^{-1}$ as a reduced expression;

(iii) $D(h_\omega) = 1$ otherwise.

Proof. Suppose $|\omega| = 1$. Then, clearly, h_ω is a homomorphism, so $D(h_\omega) = 0$. Otherwise, $\omega = \omega_1\omega_2$ as a reduced expression. In this case $h_\omega(\omega) = 1$ while $h_\omega(\omega_1) = 0$ and $h_\omega(\omega_2) = 0$. This proves the first statement.

Let $\widetilde{g}_1 = g_1 u$, $\widetilde{g}_2 = u^{-1}g_2$ and $g = \widetilde{g}_1\widetilde{g}_2 = g_1g_2$, where $g, g_1, g_2, \widetilde{g}_1, \widetilde{g}_2 \in F$, and g_1g_2 is the reduced representative of $\widetilde{g}_1\widetilde{g}_2$. Then

$$h_\omega(\widetilde{g}_1\widetilde{g}_2) - h_\omega(\widetilde{g}_1) - h_\omega(\widetilde{g}_2) = h_\omega(\widetilde{g}_1\widetilde{g}_2) - h_\omega(\widetilde{g}_1) - h_\omega(\widetilde{g}_2) -$$
$$- h_\omega(g_1) + h_\omega(g_1) - h_\omega(g_2) + h_\omega(g_2) + h_\omega(u) - h_\omega(u^{-1}) = \gamma_3 - \gamma_1 - \gamma_2,$$

where $\gamma_i \in \{0, s_i\}$ for $1 \leqslant i \leqslant 3$ and s_1, s_2, s_3 are the signs of the reduces expressions g_1u, $u^{-1}g_2$ and g_1g_2 respectively. Since we can always replace ω with ω^{-1} and reverse the order of the expressions, there are nine possibilities for the triple (s_1, s_2, s_3):

$$D(h_\omega) \leqslant \begin{cases} 0 & \text{if } (s_1, s_2, s_3) = (0,0,0); \\ 1 & \text{if } (s_1, s_2, s_3) = (1,0,0),\ (0,0,1),\ (1,-1,0) \text{ or } (1,0,1); \\ 2 & \text{if } (s_1, s_2, s_3) = (1,1,0),\ (1,1,1) \text{ or } (1,0,-1); \\ 3 & \text{if } (s_1, s_2, s_3) = (1,1,-1). \end{cases}$$

Consider first the case $(s_1, s_2, s_3) = (1, 0, -1)$. If the word ω overlaps g_1u and ω^{-1} overlaps g_1g_2, then, clearly, either some prefix of ω is equal to a substring of ω^{-1} or some prefix of ω^{-1} is equal to a substring of ω. In both cases, ω has one of the forms listed in (ii).

Suppose that (s_1, s_2, s_3) is either $(1, 1, 0)$ or $(1, 1, 1)$. Then ω overlaps both g_1u and $u^{-1}g_2$ and has the form $\omega = \omega_1\omega_2\omega_3$ where either $\omega_2\omega_3$ is the prefix of u and $\omega_1\omega_2$ is the suffix of u^{-1} or ω_3 is the prefix of u and ω_1 is the suffix of u^{-1}. In the first case, $\omega_2^{-1}\omega_1^{-1}$ is the prefix of u, so we get $\omega_2 = \omega_2^{-1}$, which is impossible. Thus, one of ω_1^{-1} and ω_3 is a prefix of the other. In both cases, ω has one of the forms listed in (ii).

Finally, consider the case $(s_1, s_2, s_3) = (1, 1, -1)$. We can assume that ω has the form $\omega = \omega_1\omega_2\omega_3\omega_2^{-1}$. without loss of generality. Here $\omega_1\omega_2\omega_3$ is the suffix of g_1 and $\omega_3\omega_2^{-1}$ is the prefix of g_2. By our hypothesis, a copy of ω^{-1} overlaps $\overline{\omega} = \omega_1\omega_2\omega_3\omega_3\omega_2^{-1}$. From Lemma 1.4.13 it follows that $\omega_3^{-1}\omega_2^{-1}\omega_1^{-1}$ does not overlap $\omega_1\omega_2\omega_3$ in $\overline{\omega}$ and the subword $\omega_2\omega_3^{-1}$ of ω^{-1} does not overlap $\omega_3\omega_2^{-1}$ in $\overline{\omega}$. Hence, the subword ω_3^{-1} of ω^{-1} does not overlap $\omega_1\omega_2\omega_3\omega_3\omega_2^{-1}$. Thus, if there is an overlap, then either the prefix

$\omega_1\omega_2$ of $\overline{\omega}$ intersects with the suffix $\omega_2^{-1}\omega_1^{-1}$ of ω^{-1} or the suffix ω_2 of $\overline{\omega}$ intersects with the prefix ω_2 of ω^{-1}. By Lemma 1.4.13 neither of these two cases can occur, so the case $(s_1, s_2, s_3) = (1, 1, -1)$ is impossible.

If the word ω has one of the forms listed in (ii), then it can be verified by example that $D(h_\omega) \geqslant 2$. For instance, if $a = b = \omega_1\omega_2\omega_1^{-1}$, then $ab = \omega_1\omega_2^2\omega_1^{-1}$ and

$$|h_\omega(ab) - h_\omega(a) - h_\omega(b)| = 2.$$

□

Definition 1.4.17. A word ω is said to be *monotone* if for each $a \in S$ at most one of a and a^{-1} appears in ω.

From the Proposition 1.4.16 it follows that $D(h_\omega) \leqslant 1$ for any reduced monotone word ω. Here $D(h_\omega) = 1$ whenever $|\omega| = 1$.

One can also study linear combinations of counting quasimorphisms. Suppose $\{\omega_i\}_{i\in I}$ is a sequence of words and $\{r_i\}_{i\in I}$ is a sequence of real numbers such that $\sum_{i\in I} |r_i|$ is finite. Then $\varphi = \sum_{i\in I} r_i h_{\omega_i}$ is a quasimorphism with $D(\varphi) \leqslant 2\sum_{i\in I} r_i$.

1.4.3 Rotation number

Rotation numbers were introduced in the study of one-dimensional dynamical systems by Poincaré [14]. Denote by $\mathrm{Homeo}(S^1)$ the group of homeomorphisms of the circle S^1. Suppose $\mathrm{Homeo}^+(S^1)$ its subgroup formed by all orientation preserving homeomorphisms in $\mathrm{Homeo}(S^1)$. Let also G be a subgroup of $\mathrm{Homeo}^+(S^1)$ and consider the covering projection $\mathbb{R} \longrightarrow S^1$. There is a universal central extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathrm{Homeo}^+(\mathbb{R})^{\mathbb{Z}} \longrightarrow \mathrm{Homeo}^+(S^1) \longrightarrow 0,$$

where $\mathrm{Homeo}^+(\mathbb{R})^{\mathbb{Z}}$ is the group of orientation preserving homeomorphisms of $\mathbb{R}$ which commute with integer translation. Equivalently, $\mathrm{Homeo}^+(\mathbb{R})^{\mathbb{Z}}$ is the group of homeomorphisms of $\mathbb{R}$ which cover homeomorphisms of S^1. Denote by $\widehat{G}$ the preimage of G in $\mathrm{Homeo}^+(\mathbb{R})^{\mathbb{Z}}$ under the covering projection. Then $\widehat{G}$ is centralized in $\mathrm{Homeo}^+(\mathbb{R})^{\mathbb{Z}}$ by the subgroup generated by a translation $x \mapsto x+1$.

Definition 1.4.18. [Poincaré's rotation number] Given an element $f \in \widehat{G}$, the *rotation number* is defined by

$$\mathrm{rot}(g) = \lim_{n\to\infty} \frac{f^{\circ n}(0)}{n}.$$

The limit is well-defined and, it is to be noted, any point can be taken instead of 0; the limit remains the same.

Lemma 1.4.19. The function $\mathrm{rot} : \widehat{G} \longrightarrow \mathbb{R}$ is a quasimorphism.

Proof. Let $t \in \mathrm{Homeo}(\mathbb{R})^{\mathbb{Z}}$ be the integer translation by one. Observe that for an element $f \in \widehat{G}$, $\mathrm{rot}(t^{\circ n} \circ f) = n + \mathrm{rot}(f)$. Now take two arbitrary elements $f, g \in \widehat{G}$ and write $f = t^{\circ n} \circ f'$ and $g = t^{\circ m} \circ g'$ where $0 \leqslant f'(0), g'(0) \leqslant 1$. It follows that $f \circ g = t^{\circ(n+m)} \circ f' \circ g'$ and

$$0 \leqslant \mathrm{rot}(f') + \mathrm{rot}(g') \leqslant 2,$$
$$0 \leqslant \mathrm{rot}(f' \circ g') \leqslant 2,$$

ans so $D(\mathrm{rot}) \leqslant 2$. □

Lemma 1.4.20. For all $r \in \mathbb{R}$ and $f, g \in \widehat{G}$, there is an inequality

$$r - 2 < [f, g](r) < r + 2.$$

Proof. Observe that by applying the integer translation by one to both f and g multiple times, we can obtain

$$q \leqslant (t^{\circ n} \circ f)(q), (t^{\circ m} \circ g)(q) < q + 1$$

for some integers n and m and any real number q. Note that this procedure does not change $[f, g]$. Thus, we can assume $q \leqslant f(q), g(q) < q + 1$. Then we obtain

$$q \leqslant f(q) \leqslant (f \circ g)(q) < f(q+1) < q+2,$$
$$q \leqslant g(q) \leqslant (g \circ f)(q) < g(q+1) < q+2.$$

Let $r = (g \circ f)(q)$, then from the second inequality we have

$$q \leqslant r < q + 2,$$

and from the first inequality we obtain

$$r - 2 < q \leqslant (f \circ g)(q) = (f \circ g \circ f^{-1} \circ g^{-1})(r) < q + 2 \leqslant r + 2.$$

Since q was arbitrary, then so is p (up to multiplication by some central element). Since any central element commutes with $[f, g] = f \circ g \circ f^{-1} \circ g^{-1}$, we get

$$r - 2 < [f, g](r) < r + 2$$

for any $r \in \mathbb{R}$. □

Lemma 1.4.20 shows that every commutator moves every point at distance at most two. Now we can use this fact to prove the following theorem which provides a connection between the rotation number and stable commutator length:

Theorem 1.4.21. Denote $H = \mathrm{Homeo}^+(\mathbb{R})^{\mathbb{Z}}$, then

$$\mathrm{scl}_H(f) = \frac{|\operatorname{rot}(f)|}{2}.$$

Proof. Consider an element $g \in H$ such that g moves some points in the positive direction and some points in the negative direction. For any point $p \in \mathbb{R}$ and sufficiently small $\varepsilon > 0$ there is a conjugate of g which translates p to $p + 1 - \frac{\varepsilon}{2}$ and there is a conjugate of g^{-1} which translates $g(p)$ to $g(p) + 1 - \frac{\varepsilon}{2}$, so there is a commutator which translates p to $p + 2 - \varepsilon$.

Consider now $f \in H$ such that $|\operatorname{rot}(f)| = r$, then $f^{\circ n}$ translates every point at a distance less than $nr + 1$. From the first part of the proof it follows that for every point $p \in \mathbb{R}$ and any $q \in \mathbb{R}$ with $|q| < 2$, there is a commutator $h \in H$ such that $h(p) = p + q$. Now we can divide the segment of length $nr + 1$ into pieces of length $2 - \varepsilon$ for sufficiently small ε and this shows that $f^{\circ n}$ can be expressed as a product of no more than $\left\lfloor \frac{nr+1}{2} \right\rfloor + 1$ commutators and an element $f' \in H$ which fixes some point. The dynamics of the orientation preserving homeomorphism f' on every interval, complementary to $\mathrm{fix}(f')$ is topologically conjugate for a translation t of the real line. That is, there exists an orientation preserving homeomorphism c such that $c(o(p, f')) = o(c(p), t)$, where $o(p, f)$ denotes the orbit of $p \in \mathbb{R}$ under the homeomorphism f. Observe that any translation of $\mathbb{R}$ is the commutator of two dilations, and therefore any such $f' \in H$ with a fixed point is a commutator. It follows that

$$\mathrm{cl}_H(f^{\circ n}) \leqslant \left\lfloor \frac{nr+1}{2} \right\rfloor + 2,$$

and so

$$\mathrm{scl}_H(f) = \lim_{n\to\infty} \frac{\mathrm{cl}_H(f^{\circ n})}{n} \leqslant \lim_{n\to\infty} \frac{\left\lfloor \frac{nr+1}{2} \right\rfloor + 2}{n} = \frac{r}{2} = \frac{|\operatorname{rot}(f)|}{2}.$$

Conversely, by Lemma 1.4.20, every commutator translates every point at a distance less than two, and since $f^{\circ n}$ translates every point at a distance less than $nr+1$, we have

$$2\,\mathrm{cl}_H(f^{\circ n}) \geqslant nr + 1,$$

and it follows that

$$\mathrm{scl}_H(f) = \lim_{n\to\infty} \frac{\mathrm{cl}_H(f^{\circ n})}{n} \geqslant \lim_{n\to\infty} \frac{nr+1}{2n} = \frac{|\operatorname{rot}(f)|}{2},$$

which finishes the proof. □

1.5 Bounded cohomology

The (co)homology theory of groups arose from both algebraic and topological sources. In this section we briefly introduce this theory.

Definition 1.5.1. Let G be a group. The *bar complex* $C_*(G)$ is the n-dimensional complex generated by n-tuples $(g_1, ..., g_n)$ with $g_i \in G$. The boundary map ∂ is defined by the formula

$$\partial(g_1, ..., g_n) = (g_2, ..., g_n) + \sum_{i=1}^{n-1} (-1)^i (g_1, ..., g_i g_{i+1}, ..., g_n) + (-1)^n (g_1, ..., g_{n-1}).$$

For a coefficient group R, define the *homology of the group* G with coefficients in R to be $H_*(C_*(G) \otimes R)$. Let $C^*(G, R) = \mathrm{Hom}(C_*(G), R)$ be the dual cochain complex and let δ denote the adjoint of ∂. The homology group of $(C^*(G, R), \delta)$ is called the *cohomology group of* G *with coefficients in* R and is denoted $H^*(G, R)$. The chain group $C_*(G)$ has a canonical basis, consisting of all n-tuples $(g_1, ..., g_n)$, $g_i \in G$, in dimension n. If R is a subgroup of $\mathbb{R}$, a cochain $\alpha \in C^n(G, R)$ is said to be *bounded* if

$$\sup |\alpha(g_1, ..., g_n)| < \infty,$$

where the supremum is taken over all n-tuples $(g_1, ..., g_n)$. This supremum is called the L^∞*-norm* of α and is denoted $\|\alpha\|_\infty$. The set of all bounded cochains forms a subcomplex $C_b(G, R)$ and its homology is the so-called *bounded cohomology of* G and is denoted $H_b^*(G)$.

Since $C_b^n(G)$ is formed by all the chains with finite norm, the norm $\|\cdot\|_\infty$ makes it into a Banach space for each n. This L^∞-norm induces a (pseudo)norm on $H_b^*(G)$ defined in a following way: given a bounded cohomology class $[\alpha] \in H_b^*(G)$, set

$$\|[\alpha]\|_\infty = \inf_\sigma \|\sigma\|_\infty,$$

where the infimum is taken over all bounded cocycles σ in the class of α. If the bounded coboundaries $B_b^n(G)$ form a closed subspace of $C_b^n(G)$, then this defines a Banach norm on $H_b^n(G)$, but, it is to be noted that it is not always the case that $B_b^n(G)$ is closed in $C_b^n(G)$.

1.5.1 Gersten boundary norm

Let us first have a look at what these definitions mean in low dimensions. Recall that $\widehat{Q}(G)$ denotes the vector space of all quasimorphisms on the group G and $Q(G)$ denotes the vector subspace of all homogeneous quasimorphisms. The defect function $D(\cdot)$ defines a pseudonorm on $\widehat{Q}(G)$ and $Q(G)$ which vanish exactly on the subspace spanned by all homomorphisms $G \longrightarrow \mathbb{R}$.

A one-dimensional cochain $\varphi \in C^1(G, \mathbb{R})$ is just a real-valued function $G \longrightarrow \mathbb{R}$, and φ is a cocycle if and only if $\delta\varphi = 0$. By the definition of the coboundary map,

$$\delta\varphi(g_1, g_2) = \varphi(g_1) + \varphi(g_2) - \varphi(g_1 g_2).$$

Hence, φ is a cocycle if and only if φ is a homomorphism and the subspace of $\widehat{Q}(G)$, spanned by homomorphisms $G \longrightarrow \mathbb{R}$, namely $\mathrm{Hom}(G, \mathbb{R})$, can be identified with $H^1(G, \mathbb{R})$. Since any nontrivial homomorphism $G \longrightarrow \mathbb{R}$ is unbounded, it immediately follows that $H_b^1(G, \mathbb{R})$ *is trivial* for any group G.

Suppose φ is a quasimorphism defined above. Then we have

$$|\delta\varphi(g_1, g_2)| = |\varphi(g_1) + \varphi(g_2) - \varphi(g_1 g_2)| \leqslant D(\varphi),$$

for any $g_1, g_2 \in G$. Hence, $\delta\varphi$ is a bounded 2-cochain, i.e. $\delta\varphi \in C_b^2(G, \mathbb{R})$ and $\|\delta\varphi\|_\infty = D(\varphi)$. Since $\delta\varphi$ is clearly a cocycle, it follows that the image of the coboundary map of a quasimorphism is a bounded 2-cocycle. For ease of notation we sometimes abbreviate $C^*(G, \mathbb{R})$ and $C_b^*(G, \mathbb{R})$ by just C^* and C_b^* respectively.

Theorem 1.5.2. There is an exact sequence

$$0 \longrightarrow H^1(G, \mathbb{R}) \longrightarrow Q(G) \xrightarrow{\delta} H_b^2(G, \mathbb{R}) \longrightarrow H^2(G, \mathbb{R}).$$

Proof. Consider the short exact sequence of cochain complexes

$$0 \longrightarrow C_b^* \longrightarrow C^* \longrightarrow C^*/C_b^* \longrightarrow 0.$$

Then there is an induced natural long exact sequence of cohomology groups

$$\ldots \longrightarrow H^n(C_b^*) \longrightarrow H^n(C^*) \longrightarrow H^n(C^*/C_b^*) \longrightarrow \ldots$$

and thus we can derive a sequence

$$H_b^1(G, \mathbb{R}) \longrightarrow H^1(G, \mathbb{R}) \longrightarrow H^1(C^*/C_b^*) \longrightarrow H_b^2(G, \mathbb{R}) \longrightarrow H^2(G, \mathbb{R}).$$

We already know that $H^1_b(G,\mathbb{R})$ is trivial. Note that a function $\varphi : G \longrightarrow \mathbb{R}$ is a quasimorphism if and only if $\delta\varphi \in C^2_b$, hence,

$$H^1(C^*/C^*_b) \cong \widehat{Q}(G)/C^1_b.$$

Observe that if φ and ψ are two quasimorphisms on G, which differ by a bounded value, then their homogenizations are the same, and thus

$$H^1(C^*/C^*_b) \cong \widehat{Q}(G)/C^1_b \cong Q(G),$$

and this finishes the proof. □

Denote the cycles and the boundaries with real coefficients by $Z_*(G)$ and $B_*(G)$ respectively. Then in dimension two, there is a short exact sequence

$$0 \longrightarrow Z_2(G) \xrightarrow{\ i\ } C_2(G) \xrightarrow{\ \partial\ } B_1(G) \longrightarrow 0.$$

It follows that $B_1(G)$ inherits a quotient norm, since $C_2(G)$ is normed and $Z_2(G)$ is its normed subspace.

Definition 1.5.3. Consider $a \in B_1(G,\mathbb{R})$. The *Gersten boundary norm* of a is defined by

$$\|a\|_B = \inf_{\partial A = a} \|A\|_1,$$

where the infimum is taken over all 2-chains $A \in C_2(G)$ with boundary a, and $\|A\|_1$ is the usual L^1-norm.

Since $B_1(G)$ is a normed space, we can identify its dual space with respect to the Gersten boundary norm.

Lemma 1.5.4. The dual space of $B_1(G)$ with respect to the Gersten boundary norm $\|\cdot\|_B$ is $\widehat{Q}(G)/H^1(G,\mathbb{R})$, and the operator norm on the dual space is $D(\cdot) = \|\delta \cdot\|_\infty$.

Proof. For a normed vector space V we denote the space of bounded linear functionals on V with the operator norm by $\overline{V}$.

Consider an element $f \in \overline{B}_1(G)$. By definition of a quotient norm, there is $F \in \overline{C}_2(G)$ such that $F(A) = f(\partial A)$ and so F vanishes on $Z_2(G)$, i.e. it is a coboundary. Thus, $F = \delta\varphi$ where $\varphi \in C^1(G)$ is unique up to some element of $H^1(G,\mathbb{R})$. It follows that φ is a quasimorphism since F is bounded. Note that f is equal to the restriction of φ to $B_1(G)$, and so we have just defined a map

$$\overline{B}_1(G) \longrightarrow \widehat{Q}(G)/H^1(G,\mathbb{R}).$$

Clearly, this map is surjective. Suppose that two different elements $f_1, f_2 \in \overline{B}_1(G)$ define the same quasimorphism $\varphi \in \widehat{Q}(G)/H^1(G,\mathbb{R})$. Then, if we restrict φ to $B_1(G)$, we get that $f_1 = f_2$, which is a contradiction, and hence the defined map is an isomorphism of vector spaces.

Pick an element $b \in B_1$ such that $\|b\|_B = 1$, so there exists a 2-chain $A \in C_2(G)$ with $\partial A = b$ and $\|A\|_1 < 1 + \varepsilon$ for some $\varepsilon > 0$. We can express A as

$$A = \sum_i r_i(g_i, h_i),$$

where $r_i \in \mathbb{R}$, $g_i, h_i \in G$, and

$$\sum_i |r_i| < 1 + \varepsilon.$$

Since $F = \delta\varphi$ and δ is adjoint to ∂, we have

$$\begin{aligned}\frac{|F(A)|}{1+\varepsilon} \leqslant \sup_i |F(g_i,h_i)| = \sup_i |\delta\varphi(g_i,h_i)| = \sup_i |\varphi(\partial(g_i,h_i))| = \\ = \sup_i |\varphi(g_i h_i) - \varphi(g_i) - \varphi(h_i)| \leqslant D(\varphi).\end{aligned}$$

It follows that the operator norm of F does not exceed $D(\varphi)$.

On the other hand, suppose g_1 and g_2 are two arbitrary elements of G such that $g_1, g_2 \neq e$. Then $\partial(g_1, g_2) = g_1 + g_2 - g_1 g_2$ and $\|\partial(g_1, g_2)\|_1 = 3$, so we obtain

$$1 \geqslant \|\partial(g_1,g_2)\|_B \geqslant \frac{1}{3}\|\partial(g_1,g_2)\|_1 = 1,$$

but we have $F(g_1, g_2) = \varphi(g_1) + \varphi(g_2) - \varphi(g_1, g_2)$, and so there exist $g_1, g_2 \in G$ such that $\|\partial(g_1, g_2)\|_B = 1$ for which the value $|F(g_1, g_2)|$ is arbitrary close to the value of defect of the quasimorphism φ. Hence, the operator norm if F is at least equal to $D(\varphi)$ and overall, the operator norm on the dual space is equal to $D(\cdot) = \|\delta \cdot \|_\infty$. □

The dual space of a normed vector space is always a Banach space, thus the space $\widehat{Q}(G)/H^1(G,\mathbb{R})$ is a Banach space. Now $Q(G)/H^1(G,\mathbb{R})$ is a Banach space since it is a closed subspace of $\widehat{Q}(G)/H^1(G,\mathbb{R})$, since the closed subspace of a Banach space is always a Banach space.

Lemma 1.5.5. Let φ be a homogeneous quasimorphism on a group G. Then

$$D(\varphi) = \|[\delta\varphi]\|_\infty \geqslant \frac{1}{2}D(\varphi).$$

Proof. By definition,

$$\|[\delta\varphi]\|_\infty = \inf_\sigma \|\sigma\|_\infty,$$

where the infimum is taken over all bounded 2-cocycles σ in the cohomology class of $\delta\varphi$. Any such σ is of the form $\delta\psi$ where ψ is some unique quasimorphism, such that $\psi - \varphi \in C_b^1(G)$. In particular, φ is a homogenization of ψ and hence we have an inequality

$$\|[\delta\varphi]\|_\infty = \inf_\psi D(\psi) \leqslant D(\varphi),$$

where infimum is taken over all quasimorphisms ψ such that $\psi - \varphi \in C_b^1(G)$. It is actually suffices to take the infimum over antisymmetric quasimorphisms ψ, since antisymmetrization does not increase the defect.

Pick two elements $g_1, g_2 \in G$ such that $|\delta\varphi(g_1, g_2)|$ is very close to the value $D(\varphi)$. From the proof of Lemma 1.4.10 we know that an element $g_1^{2n} g_2^{2n} (g_1 g_2)^{-2n}$ can be expressed as a product of at most n commutators. From Lemma 1.4.8 it follows that

$$|\psi(g_1^{2n} g_2^{2n} (g_1 g_2)^{-2n})| \leqslant (4n-1)D(\psi),$$

since ψ is antisymmetric. Since $\psi - \varphi \in C_b^1(G)$, then there is a constant $C \geqslant 0$ such that

$$|\psi(g_1^{2n} g_2^{2n} (g_1 g_2)^{-2n}) - \varphi(g_1^{2n} g_2^{2n} (g_1 g_2)^{-2n})| \leqslant C.$$

Note that the constant C does not depend on g_1, g_2 and n. Moreover, since φ is homogeneous, we have

$$\begin{aligned} |\varphi(g_1^{2n} g_2^{2n} (g_1 g_2)^{-2n}) - 2n\varphi(g_1) - 2n\varphi(g_2) + 2n\varphi(g_1 g_2)| &= \\ |\varphi(g_1^{2n} g_2^{2n} (g_1 g_2)^{-2n}) - 2n \cdot \delta\varphi(g_1, g_2)| &\leqslant 2D(\varphi), \end{aligned}$$

and so

$$\lim_{n\to\infty} \frac{\varphi(g_1^{2n} g_2^{2n} (g_1 g_2)^{-2n})}{2n} = |\delta\varphi(g_1, g_2)|.$$

Recall that $|\delta\varphi(g_1, g_2)|$ is arbitrary close to $D(\varphi)$. Combining everything together, we get

$$D(\varphi) \leqslant 2D(\psi),$$

which proves the lemma. □

Typically, the Banach space $Q(G)/H^1(G, \mathbb{R})$ is very big, even for finitely presented groups.

Example 1.5.6. Let $F = F\langle a, b\rangle$ be a free group on two generators. For each positive n consider a word $\omega_n = ab^n a$ and for each map $f : \mathbb{Z}_+ \longrightarrow \{0, 1\}$ define

$$\overline{H}_f = \sum_n f(n)\overline{H}_{\omega_n},$$

where each $\overline{H}_{\omega_n}$ denotes the homogenization of the big counting quasimorphism. Since the words do not overlap, we have $D(H_f) = 1$ and by Lemma 1.5.5, $D(\overline{H}_f) \leqslant 2$. If f, g are two different functions $\mathbb{Z}_+ \longrightarrow \{0, 1\}$ and n is in the support of f and not in the support of g, then

$$(\overline{H}_f - \overline{H}_g)(ab^n a) = 1.$$

On the other hand, both $\overline{H}_f$ and $\overline{H}_g$ are not in $H^1(G, \mathbb{R})$, since $\overline{H}_f(a) = \overline{H}_g(a) = 0$. It follows that $D(\overline{H}_f - \overline{H}_g) > 0$, and since both functions are taking integer values, $D(\overline{H}_f - \overline{H}_g) \geqslant 1$. In words, we have just constructed a subset of $Q(G)/H^1(G, \mathbb{R})$ of cardinality $2^{\aleph_0}$.

1.6 Bavard duality

In this section we aim to prove Bavard's Duality Theorem, which provides a functional analysis characterization of stable commutator length.

Lemma 1.6.1. Let X be a normed space with norm $\|\cdot\|_X$ and let $\overline{X}$ be its dual space with norm $\|\cdot\|_{\overline{X}}$. Let also $x_0 \in X$ be such that $x_0 \neq 0$. Then there exists a bounded linear functional $\widetilde{f}$ on X such that $\|\widetilde{f}\|_{\overline{X}} = 1$ and $\widetilde{f}(x_0) = \|x_0\|_X$.

Proof. We aim to find a subspace $Z \subset X$, containing x_0 and a linear functional $f \in \overline{Z}$ such that $f(x_0) = \|x_0\|_Z$ and $\|f\|_{\overline{Z}} = 1$. For instance, we can take

$$Z = \{ax_0 \mid a \in \mathbb{R}\},$$

i.e. the one-dimensional subspace of X spanned by x_0. Let the functional $f : Z \longrightarrow \mathbb{R}$ be defined by

$$f(ax_0) = a\|x_0\|_X.$$

Then, according to the Hahn-Banach Theorem, there exists and element of the dual space $\widetilde{f} \in \overline{X}$ such that $\widetilde{f}(x_0) = f(x_0) = \|x_0\|_X$ and $\|\widetilde{f}\|_{\overline{X}} = \|f\|_{\overline{Z}} = 1$, as required. □

Corollary 1.6.2. For each element x in a normed space X we have

$$\|x\|_X = \sup \frac{f(x)}{\|f\|_{\overline{X}}},$$

where the supremum is taken over all nonzero linear functionals in the dual space $\overline{X}$.

Proof. The lemma implies that there is some functional $f \in \overline{X}$ with norm equal to one and which takes $x \in X$ to $\|x\|_X$. It follows that

$$\sup \frac{|f(x)|}{\|f\|_{\overline{X}}} \geqslant \|x\|_X.$$

On the other hand, since f is bounded, we have

$$|f(x)| \leqslant \|f\|_{\overline{X}} \cdot \|x\|_X,$$

and it follows that

$$\frac{|f(x)|}{\|f\|_{\overline{X}}} \leqslant \|x\|_X,$$

and this implies the desired equality. □

Suppose now that G is a group and $a \in [G,G]$ so that $a \in B_1(G)$ as a cycle. Then, according to Lemma 1.5.4 and Corollary 1.6.2, we have an equality:

$$\|a\|_B = \sup_{\varphi \in \overline{B}_1(G)} \frac{|\varphi(a)|}{D(\varphi)},$$

where $\overline{B}_1(G) = \widehat{Q}(G)/H^1(G,\mathbb{R})$.

Now we can relate the Gersten boundary norm to stable commutator length.

Definition 1.6.3. Let G be a group and $a \in [G,G]$ so that $a \in B_1(G)$ as a cycle. Define the *filling norm* of a, denoted $\mathrm{fill}(a)$ to be the homogenization of $\|a\|_B$. That is,

$$\mathrm{fill}(a) = \lim_{n\to\infty} \frac{\|a^n\|_B}{n}.$$

For any n, m there is an identity

$$\partial(a^n, a^m) = a^n + a^m - a^{n+m},$$

and so we have

$$\|a^{n+m}\|_B \leqslant \|a^n\|_B + \|a^m\|_B + 1.$$

The following lemma shows that the limit in Definition 1.6.3 exists.

Lemma 1.6.4. Let $\{x_i\}_{n=1}^{\infty}$ be a sequence of real numbers with each $x_i \geqslant 0$ and such that $x_{n+m} \leqslant x_n + x_m + 1$ for any n, m. Then

$$\lim_{n\to\infty} \frac{x_n}{n} \in \mathbb{R} \cup \{-\infty\}.$$

Proof. Suppose that

$$\liminf_{n\to\infty} \frac{x_n}{n} < b < c.$$

Then there exists $n > \frac{2}{c-b}$ such that $\frac{x_n}{n} < b$. For sufficiently large $l > n$, so that $l(c-b) > 2\max_{r<n} x_n$, write $l = nk + r$, where $0 < r < n$, and so

$$\frac{x_l}{l} \leqslant \frac{kx_n + x_r + k}{l} \leqslant \frac{x_n}{n} + \frac{x_r}{l} + \frac{1}{n} \leqslant b + \frac{c-b}{2} + \frac{c-b}{2} = c.$$

Hence we have

$$\limsup_{n\to\infty} \frac{x_n}{n} = \liminf_{n\to\infty} \frac{x_n}{n},$$

and so the limit exists. □

Now we are able to provide a relation between the Gersten boundary norm and stable commutator length.

Lemma 1.6.5. [Bavard] Let G be a group and let $a \in [G, G]$. There is an equality

$$\mathrm{scl}_G(a) = \frac{1}{4}\,\mathrm{fill}(a).$$

Proof. Since a^n can be expressed as a product of commutators, we are able to construct an orientable surface S with one boundary component together with a homomorphism $\varphi : \pi_1(S) \longrightarrow G$ such that

$$\varphi_* \partial S = a^n$$

in $\pi_1(S)$. Suppose S has genus g. We can construct a one-vertex triangulation of S with $4g - 1$ triangles, where one edge maps to the boundary, and thus

$$\|a^n\|_B \leqslant 4\,\mathrm{cl}_G(a^n) - 1.$$

Dividing both sides by n and taking the limit as $n \to \infty$, we get

$$\mathrm{fill}(a) \leqslant 4\,\mathrm{scl}_G(a).$$

Conversely, let X be a topological space with $\pi_1(X) = G$ and let $\gamma : S^1 \longrightarrow X$ be a loop, representing the conjugacy class of a. Now let A be a chain such that $\partial A = a$ with the L^1-norm $\|a\|_1$ close to $\|a\|_B$. Let V be the finite dimensional subspace of $C_2(G, \mathbb{R})$ of the 2-chains with the support contained in the support of A. Note that V is a rational subspace, and therefore since a is a rational chain, the subspace $\partial^{-1}(a) \cap V$ rationally "approximates" the chain A, i.e. it contains rational points arbitrarily close to A. Then by changing A in such a way that its norm is altered by an arbitrarily small amount, we may assume that A is rational. Furthermore, after scaling by some integer, we may assume that A is integral with $\partial A = na$ for which the value $\frac{\|A\|_1}{n\|a\|_B}$ is very close to one. Write $A = \sum_i n_i \sigma_i$, where each $n_i \in \mathbb{Z}$ and each σ_i is a singular 2-simplex, i.e. a map $\sigma_i : \Delta^2 \longrightarrow X$. We can group the edges of σ_i's in pairs except for those edges mapping to γ. The result of such a pairing is an orientable surface S with a map $\varphi : S \longrightarrow X$ such that $\varphi_*(A_S) = A$, where A_S is a chain representing the fundamental class $[S]$. By construction we have $\|A_S\|_1 = \|A\|_1$, and so using Theorem A.4 and Proposition 1.3.1 we obtain

$$\frac{\|A\|_1}{n} = \frac{\|A_S\|_1}{n} \geqslant \frac{-2\chi(S)}{n} \geqslant 4 \cdot \frac{-\chi^-(S)}{2n} = 4\,\mathrm{scl}_G(a).$$

Since we can take $\frac{\|A_S\|_1}{n}$ to be arbitrarily close to $\|a\|_B$, we can then apply a homogenization to get

$$\mathrm{fill}(a) \geqslant 4\,\mathrm{scl}_G(a),$$

which finishes the proof. □

Finally, we have all the preliminary results to be able to prove Bavard's Duality Theorem.

Theorem 1.6.6. [Bavard's Duality Theorem] Let G be a group. If the quotient space $Q_H(G) = Q(G)/H^1(G,\mathbb{R})$ is not trivial, then for any element $a \in [G,G]$ there is an equality

$$\mathrm{scl}_G(a) = \frac{1}{2} \sup_{\varphi \in Q_H(G)} \frac{|\varphi(a)|}{D(\varphi)}. \tag{1.4}$$

In case when $Q_H(G)$ is trivial, $\mathrm{scl}_G(a) = 0$ for all $a \in [G,G]$.

Proof. As we already know, Corollary 1.6.2 implies the following equality:

$$\|a\|_B = \sup_{\varphi \in \overline{B}_1(G)} \frac{|\varphi(a)|}{D(\varphi)},$$

where $\overline{B}_1(G) = \widehat{Q}(G)/H^1(G,\mathbb{R})$. Homogenization of both sides combined with Lemma 1.6.5 gives us

$$\mathrm{scl}_G(a) = \frac{1}{4} \lim_{n\to\infty} \left(\sup_{\varphi \in \overline{B}_1(G)} \frac{|\varphi(a^n)|}{nD(\varphi)} \right).$$

Let $\widehat{\varphi}$ to be the homogenization of φ. By Lemma 1.4.4, there is an estimate

$$|\varphi(a^n) - \widehat{\varphi}(a^n)| \leqslant D(\varphi),$$

from which it follows that for each n and any quasimorphism φ,

$$\frac{|\varphi(a^n) - \widehat{\varphi}(a^n)|}{nD(\varphi)} \leqslant \frac{1}{n}.$$

Note that if $Q_H(G)$ is trivial, then $\widehat{\varphi} = 0$, and so

$$\frac{|\varphi(a^n)|}{nD(\varphi)} \leqslant \frac{1}{n}.$$

It follows that in this case stable commutator length vanishes.

For each n, consider the sequence $\{\varphi_{n,i}\}$ of elements in $\widehat{Q}(G)$ such that

$$\left| \frac{\varphi_{n,m}(a^n)}{nD(\varphi_{n,m})} - \sup_{\varphi \in \overline{B}_1(G)} \frac{|\varphi(a^n)|}{nD(\varphi)} \right| \leqslant \frac{1}{m},$$

then

$$\left| \frac{\widehat{\varphi}_{n,m}(a^n)}{nD(\varphi_{n,m})} - \sup_{\varphi \in \overline{B}_1(G)} \frac{|\varphi(a^n)|}{nD(\varphi)} \right| =$$

$$= \left| \frac{\widehat{\varphi}_{n,m}(a^n)}{nD(\varphi_{n,m})} - \frac{\varphi_{n,m}(a^n)}{nD(\varphi_{n,m})} + \frac{\varphi_{n,m}(a^n)}{nD(\varphi_{n,m})} - \sup_{\varphi \in \overline{B}_1(G)} \frac{|\varphi(a^n)|}{nD(\varphi)} \right| \leqslant \frac{1}{n} + \frac{1}{m}.$$

Consider a diagonal subsequence $\{\varphi_{n,n}\}$. We get

$$\left| \frac{\widehat{\varphi}_{n,n}(a^n)}{nD(\varphi_{n,n})} - \sup_{\varphi \in \overline{B}_1(G)} \frac{|\varphi(a^n)|}{nD(\varphi)} \right| \leqslant \frac{2}{n},$$

and hence we obtain

$$\mathrm{scl}_G(a) = \frac{1}{4} \sup_{\varphi \in \overline{B}_1(G)} \frac{|\widehat{\varphi}(a^n)|}{nD(\varphi)}.$$

Since $\widehat{\varphi}$ is homogeneous,

$$\mathrm{scl}_G(a) = \frac{1}{4} \sup_{\varphi \in \overline{B}_1(G)} \frac{|\widehat{\varphi}(a)|}{D(\varphi)}.$$

From the proof of Lemma 1.5.5 it follows that

$$D(\varphi) \geqslant \frac{1}{2} D(\widehat{\varphi}),$$

so we get

$$\mathrm{scl}_G(a) = \frac{1}{4} \sup_{\varphi \in \overline{B}_1(G)} \frac{|\widehat{\varphi}(a)|}{D(\varphi)} \leqslant \frac{1}{4} \sup_{\varphi \in \overline{B}_1(G)} \frac{2|\widehat{\varphi}(a)|}{D(\widehat{\varphi})} = \frac{1}{2} \sup_{\varphi \in Q_H(G)} \frac{|\varphi(a)|}{D(\varphi)}.$$

Conversely, suppose φ is a homogeneous quasimorphism and a^n can be written as a product of m commutators. Then, by Lemma 1.4.9 we have

$$|\varphi(a^n)| \leqslant 2mD(\varphi),$$

so

$$\mathrm{cl}_G(a^n) \geqslant \frac{1}{2} \frac{|\varphi(a^n)|}{D(\varphi)}.$$

Dividing both sides by n and taking the limit as $n \to \infty$, we get

$$\mathrm{scl}_G(a) \geqslant \frac{1}{2} \sup_{\varphi \in Q_H(G)} \frac{|\varphi(a)|}{D(\varphi)},$$

which proves the equality (1.4). □

1.7 Amenable groups

The behaviour of stable commutator length can be different, depending on the group. In this section we aim to show that the function scl_G vanishes in case when G is *amenable*.

We briefly present the theory of amenability following A.L.T. Paterson [13]. The study of amenable groups arose from the study of finitely additive, invariant measure theory. The concept of isometry-invariant measure leads to the well-known Banach-Tarski paradox and the theory of paradoxical decompositions. In 1929, a few years later after the Banach-Tarski Theorem, von Neumann introduced the class of amenable groups and used them to explain why the paradox occurs only in dimension at least three, however, the term *amenable* was introduced by M.M. Day as a pun a couple of decades later, in 1950.

Definition 1.7.1. Let G be a locally compact topological group. A *measure* on G is a finitely additive measure μ on the subsets of G, such that $\mu(G) = 1$, which is left-invariant; that is, for all $g \in G$ and $S \subseteq G$, $\mu(gS) = \mu(S)$. We say that G is *amenable* if it has such a measure.

There are many equivalent conditions for amenability. For instance, any amenable group G has a *fixed point property*; that is, any linear action of the amenable group G on a compact convex subset of a (separable) locally convex topological vector space has a fixed point. It is easy to see that any finite group is amenable. Furthermore, any soluble group is amenable.

Proposition 1.7.2. Let G be amenable. Then every homogeneous quasimorphism $\varphi : G \longrightarrow \mathbb{R}$ is a homomorphism.

Proof. Let $\varphi : G \longrightarrow \mathbb{R}$ be a quasimorphism. Note that to prove the proposition, it is enough to show that there exists a homomorphism which differs from φ by a bounded amount. Denote by $\mathbb{R}^{G\times G}$ the space of functions $G \times G \longrightarrow \mathbb{R}$ with the topology of pointwise convergence. We can define a function $\Phi : G \times G \longrightarrow \mathbb{R}$ by

$$\Phi(a, b) = \varphi(a) - \varphi(b).$$

The group G acts on $G \times G$ diagonally; that is, for all $g \in G$ and for all $(a, b) \in G \times G$,

$$g(a, b) = (ga, gb).$$

Hence, G acts diagonally on $\mathbb{R}^{G\times G}$. For any $g \in G$ we have

$$g\Phi(a,b) = \varphi(ga) - \varphi(gb),$$

and so

$$|g\Phi(a,b) - \Phi(a,b)| = |\varphi(ga) - \varphi(gb) - \varphi(a) + \varphi(b)| = \\ |\varphi(ga) - \varphi(g) - \varphi(a) - (\varphi(gb) - \varphi(g) - \varphi(b))| \leqslant 2D(\varphi).$$

So, the set

$$B = \mathrm{conv}(G\Phi),$$

which is a convex hull of the orbit $G\Phi$ is a compact, convex subset in $\mathbb{R}^{G\times G}$. Note that for any $a, b, c \in G$,

$$\Phi(a,b) + \Phi(b,c) = \varphi(a) - \varphi(b) + \varphi(b) - \varphi(c) = \Phi(a,c),$$

and for any $a \in G$, $\Phi(a,a) = 0$. Furthermore, Φ is antisymmetric in its arguments. This property is invariant under the action of G, and preserved under limits and linear combinations, so it holds for any element of the convex hull B.

Since G is amenable, any linear action of G, which is invariant on a compact convex subset of a locally compact convex topological vector space, has a fixed point in this set. Let $\Psi : G \times G \longrightarrow \mathbb{R}$ be such a G-invariant function. Define $\psi : G \longrightarrow \mathbb{R}$ by

$$\psi(a) = \Psi(a, e_G),$$

where $e_G \in G$ is a group identity. Since Ψ is G-invariant, we have

$$\psi(ab) = \Psi(ab, e_G) = a\Psi(b, a^{-1}) = \Psi(b, a^{-1}).$$

We also know that

$$\Psi(b, a^{-1}) + \Psi(a^{-1}, e_G) = \Psi(b, e_G),$$

so

$$\psi(ab) = \psi(b) - \psi(a^{-1}).$$

We also know that Ψ is antisymmetric in its arguments, thus

$$\psi(a^{-1}) = \Psi(a^{-1}, e_G) = \Psi(e_G, a) = -\Psi(a, e_G) = -\psi(a),$$

and hence we have

$$\psi(ab) = \psi(a) + \psi(b),$$

which finishes the proof. □

Consider the space $\widehat{Q}(G)$ of all quasimorphisms on G and its subspaces $\widehat{Q}_0(G)$ and $\widehat{Q}_1(B) \cong H^1(G, \mathbb{R})$, consisting of bounded functions and homomorphisms respectively. Note that $\widehat{Q}_0(G) \cap \widehat{Q}_1(G)$ is trivial. Consider the quotient spaces

$$\widehat{R}(G) = \widehat{Q}(G)/\widehat{Q}_0(G)$$

and

$$R(G) = \widehat{Q}(G)/(\widehat{Q}_0(G) \oplus \widehat{Q}_1(G)) \cong \widehat{R}(G)/H^1(G, \mathbb{R}).$$

For any element $\varphi \in \widehat{Q}(G)$ there is a homogenization $\widehat{\varphi} \in Q(G)$, and hence, since any two quasimorphisms, which differ by a bounded amount, have the same homogenization, we have

$$\widehat{R}(G) \cong Q(G),$$

and therefore

$$R(G) \cong Q(G)/H^1(G, \mathbb{R}).$$

Since G is amenable, Proposition 1.7.2 tells us precisely that $R(G)$ is trivial.

Theorem 1.7.3. Let G be amenable. Then for all $g \in [G, G]$,

$$\mathrm{scl}_G(g) = 0.$$

Proof. Since $Q(G)/H^1(G, \mathbb{R})$ is trivial, the statement of the theorem directly follows from Bavard's Duality Theorem (Theorem 1.6.6). □

1.8 General method for the vanishing theorems

Some proofs of the fact that stable commutator length vanishes on certain class of groups actually proceed by showing that the commutator length in this class of groups is bounded. However, D. Kotschick [9] emphasized that vanishing stable commutator length is a weaker conclusion, so it may be proved with less effort.

Proposition 1.8.1. Let G be a group and let $H < G$ be a subgroup with the property that there is an arbitrary large number of conjugate embeddings $H_i < G$ of H in G such that elements of H_i and H_j commute in G whenever $i \neq j$. Then every homogeneous quasimorphism on G restricts to H as a homomorphism.

Proof. First of all we show that $\varphi([a,b]) = 0$ for all $a, b \in H$. Denote by $a_i, b_i \in G$ the images of a and b under the embedding map for $H_i < G$. Since φ is homogeneous, it is constant on conjugacy classes, and so

$$n\varphi([a,b]) = \varphi([a_1, b_1]) + ... + \varphi([a_n, b_n]).$$

Applying Lemma 1.4.5 to the right hand side, we get

$$n\varphi([a,b]) = \varphi([a_1, b_1] \cdot ... \cdot [a_n, b_n]),$$

and since elements of the different embeddings H_i and H_j commute, we obtain

$$n\varphi([a,b]) = \varphi([a_1...a_n, b_1...b_n]),$$

but $\varphi([a_1...a_n, b_1...b_n]) \leqslant D(\varphi)$ and so the same bound holds for the left hand side. If $\varphi([a,b]) \neq 0$, then the value of $n\varphi([a,b])$ is unbounded, since we can take n to be arbitrarily large, and the value of $\varphi([a_1...a_n, b_1...b_n])$ is still bounded, so we have a contradiction and $\varphi([a,b]) = 0$.

From Lemma 1.4.10 we know that the element $a^{2n}b^{2n}(ab)^{-2n}$ can be expressed as a product of at most n commutators. Then so is $(ab)^{2n}a^{-2n}b^{-2n}$. From the proof of Lemma 1.4.5 we know that

$$|\varphi(ab) - \varphi(a) - \varphi(b)| = \frac{1}{2n} \lim_{n\to\infty} |\varphi((ab)^{2n}a^{-2n}b^{-2n})|.$$

It follows that the right hand side is bounded above by $\frac{1}{2}D(\varphi)$. By taking the supremum of the left hand side, we get $D(\varphi) \leqslant \frac{1}{2}D(\varphi)$, so φ is a homomorphism. □

Recall that a group G is said to be *perfect*, if it equals its own derived subgroup, i.e. $G = [G, G]$.

Theorem 1.8.2. Let G be a group in which every element can be expressed as a product of some fixed number of elements contained in distinguished subgroups $H < G$. If each subgroup H is perfect and has the property that there is an arbitrary large number of conjugate embeddings $H_i < G$ of H in G such that elements of H_i and H_j commute in G whenever $i \neq j$, then for any $g \in [G, G]$,

$$\mathrm{scl}_G(g) = 0.$$

Proof. Suppose G is a group in which every element can be expressed as a product of k elements contained in distinguished subgroups $H < G$. Since each subgroup H is perfect, from the proof of Proposition 1.8.1 it follows that the restriction of any homogeneous quasimorphism $\varphi : G \longrightarrow \mathbb{R}$ to H vanishes. Therefore the value of $\varphi(g)$ for any $g \in G$ is bounded by $(k-1)D(\varphi)$, and every bounded quasimorphism is trivial, hence we are done. □

1.9 Stable commutator length as a norm

Stable commutator length can be naturally generalized to a pseudonorm on some quotient of $B_1(G)$. In this case, Bavard duality holds in the broader context with essentially the same proof. First of all, we show how the functions cl_G and scl_G can be extended to finite sums.

Definition 1.9.1. Let G be a group and let $g_1, ..., g_m$ be not necessarily distinct elements of G. If the product $g_1 \cdot ... \cdot g_m \in [G, G]$, then define $\mathrm{cl}_G(g_1 + ... + g_m)$ to be the least number of commutators whose product is equal to an expression of the form

$$g_1 h_1 g_2 h_1^{-1} h_2 g_2 h_2^{-1} ... h_{m-1} g_m h_{m-1}^{-1}$$

for some elements $h_i \in G$, $1 \leqslant i \leqslant m-1$. In other words,

$$\mathrm{cl}_G(g_1 + ... + g_m) = \inf \mathrm{cl}_G(g_1 h_1 g_2 h_1^{-1} h_2 g_2 h_2^{-1} ... h_{m-1} g_m h_{m-1}^{-1}),$$

where the infimum is taken over all $h_1, ..., h_{m-1} \in G$. Then define

$$\mathrm{scl}_G(g_1 + ... + g_m) = \lim_{n \to \infty} \frac{\mathrm{cl}_G(g_1^n + ... + g_m^n)}{n}.$$

Note that cl_G and scl_G depend only on the individual conjugacy classes of the summands, and are commutative in their arguments.

There is the following geometrical interpretation of this generalization. If X is a topological space such that $\pi_1(X) = G$ and $\gamma_1, ..., \gamma_m$ are loops which represent the conjugacy classes of $g_1, ..., g_m$ correspondingly, then $\mathrm{cl}_G(g_1 + ... + g_m)$ is the smallest genus of a surface S with m boundary components ∂_i for which there is a map $f : S \longrightarrow X$ wrapping each ∂_i around γ_i.

The following lemma shows that the limit in Definition 1.9.1 exists.

Lemma 1.9.2. Suppose the product $g_1, ..., g_m \in [G, G]$. Then the limit

$$\lim_{n \to \infty} \frac{\mathrm{cl}_G(g_1^n + ... + g_m^n)}{n}$$

exists.

Proof. Note that the function $\mathrm{cl}_{G,n}$ defined by

$$\mathrm{cl}_{G,n}(g_1 + ... + g_m) = \mathrm{cl}_G(g_1^n + ... + g_m^n)$$

is not subadditive, but the "corrected" function $\mathrm{cl}_{G,n,m}$, defined by

$$\mathrm{cl}_{G,n,m}(g_1 + ... + g_m) = \mathrm{cl}_G(g_1^n + ... + g_m^n) + (m-1)$$

is actually subadditive. Suppose that S_n and S_k are two surfaces with m boundary components, each of which wraps n and k times respectively around each of the m loops. Then these two surfaces can be connected together via m rectangles. This operation gives us a surface S of genus $g_{S_n} + g_{S_k} + (m-1)$ with m boundary components, each of which wraps $n+k$ times around each of the m loops. Here g_{S_n} and g_{S_k} denote the genus of S_n and S_k respectively. Thus, by Fekete's Lemma the limit on the left hand side exists, and then so is the limit on the right hand side:

$$\lim_{n\to\infty} \frac{\mathrm{cl}_G(g_1^n + ... + g_m^n) + (m-1)}{n} = \lim_{n\to\infty} \frac{\mathrm{cl}_G(g_1^n + ... + g_m^n)}{n}.$$

□

Let S be a compact, connected, oriented surface. Given a group G, a topological space X with $\pi(X) = G$ and loops $\gamma_j : S^1 \longrightarrow X$, $1 \leqslant j \leqslant m$, we call a map $f : S \longrightarrow X$ *admissible* if there is a commutative diagram

$$\begin{array}{ccc} \partial S & \xrightarrow{\quad i \quad} & S \\ {\scriptstyle \partial f}\downarrow & & \downarrow{\scriptstyle f} \\ \coprod_j S^1 & \xrightarrow[\coprod_j \gamma_j]{} & X \end{array}$$

where $i : \partial S \longrightarrow S$ is the inclusion map. Define $n(S)$ by the identity

$$\partial f_*[\partial S] = n(S) \left[\coprod_j S^1 \right],$$

as before. Informally, $n(S)$ is the common degree with which ∂S wraps around each of the m loops γ_j.

Proposition 1.9.3. Given a group G and a topological space X with $\pi_1(X) = G$. For $1 \leqslant j \leqslant m$, let $\gamma_j : S^1 \longrightarrow X$ be a loop representing the conjugacy class of an element $a_j \in G$. Then

$$\mathrm{scl}_G(a_1 + ... + a_m) = \inf_S \frac{-\chi^-(S)}{2n(S)}, \tag{1.5}$$

where the infimum is taken over all admissible maps $f : S \longrightarrow X$ as defined above.

Proof. Note that the proof is almost the same as that of Proposition 1.3.1. The inequality in one direction follows from the definition, if we use the function $\mathrm{cl}_{G,n,m}$ instead of $\mathrm{cl}_{G,n}$:

$$\mathrm{scl}_G(a_1 + ... + a_m) \geqslant \inf_S \frac{-\chi^-(S)}{2n(S)}.$$

Conversely, let S be a surface with admissible map $f : S \longrightarrow X$. We may assume without loss of generality that each component of S_i has at least one boundary component which maps to some γ_i with some nontrivial degree. Suppose N is a sufficiently big even fixed integer. For each S_i we can construct a cover C_i of degree $2N$ such that C_i is connected and has at most as many boundary components as S_i. As before, by gluing on a constant number of rectangles, each C_i can be modified to have exactly m boundary components, each mapping to some γ_i with degree $2Nn(S)$. This operation raises the value $-\chi^-(S)$ by an amount independent of N, so the inequality in the reverse direction follows. □

A surface S is said to be *extremal* if it realizes the infimum in (1.5). It is easy to see that if such extremal surface exists, then scl_G must be rational.

The generalized function scl_G defined above can be easily extended to integral group 1-chains. Suppose that n is any nonnegative integer and $a, a_i \in G$, $1 \leqslant i \leqslant m$. Let X be a topological space with $\pi_1(X) = G$, and let $\gamma : S^1 \longrightarrow X$ be a loop representing the conjugacy class of a. Now let S be a surface, which maps to X in such a way that its n boundary components wrap around γ a total of m times for sufficiently large m, and the rest wrap around loops γ_i representing the conjugacy classes of a_i. We can modify S by tubing together the different boundary components wrapping around γ. This operation increases the value $-\chi^-(S)$ by $n-1$. Note that m can be arbitrarily big in comparison to n.

On the other hand, suppose that one of the boundary components of a surface S mapping to X wraps around γ^n with some degree, and the rest boundary components wrap around the loops γ_i, $1 \leqslant i \leqslant m$. Then we can take n copies of S as a surface. This discussion suggests us that the generalized function scl_G satisfies

$$\mathrm{scl}_G\left(a^n + \sum_{i=1}^m a_i\right) = \mathrm{scl}_G\left(\underbrace{a + ... + a}_{n} + \sum_{i=1}^m a_i\right).$$

Similarly, suppose that X is a topological space with $\pi_1(X) = G$, and γ, γ_i are the loops representing the conjugacy classes of a and a_i respectively, $1 \leqslant i \leqslant m$. Let S be a surface with boundary components wrapping around γ_i. Take a surface S' to be the disjoint union of a surface S with some number of parallel copies of an annulus from γ to γ^{-1}. Then it is easy to see that $-\chi^-(S') = -\chi^-(S)$.

Conversely, if a surface S has one boundary component which bounds γ^m and one boundary component which bounds γ^{-m}, then we can glue these two boundary components to obtain a surface S' with $-\chi^-(S') = -\chi^-(S)$. It follows that

$$\operatorname{scl}_G\left(a + a^{-1} + \sum_{i=1}^{m} a_i\right) = \operatorname{scl}_G\left(\sum_{i=1}^{m} a_i\right).$$

We conclude that for any elements $a, a_1, ..., a_m \in G$ and for any equality of the form $n = n_1 + ... + n_k$ over $\mathbb{Z}$, there is an identity

$$\operatorname{scl}_G\left(a^n + \sum_{i=1}^{m} a_i\right) = \operatorname{scl}_G\left(\sum_{j=1}^{k} a^{n_j} + \sum_{i=1}^{m} a_i\right).$$

Furthermore, for any integer n we have

$$|n| \operatorname{scl}_G\left(\sum_{i=1}^{m} a_i\right) = \operatorname{scl}_G\left(\sum_{i=1}^{m} n a_i\right) = \operatorname{scl}_G\left(\sum_{i=1}^{m} a_i^n\right).$$

In other words, for any integral group 1-chain we have

$$\operatorname{scl}_G\left(\sum_i n_i g_i\right) = \operatorname{scl}_G\left(\sum_i g_i^{n_i}\right).$$

Observe that the result is subadditive under addition of chains, so scl_G can be extended to rational chains by linearity and to real chains by continuity, so it extends in a unique way to a pseudonorm on the real vector space $B_1(G)$.

Recall that in section 1.5.1 we defined the Gersten boundary norm $\|\cdot\|_B$ on $B_1(G)$ by

$$\|a\|_B = \inf_{\partial A = a} \|A\|_1,$$

where $a \in B_1(G)$ and the infimum is taken over all 2-chains $A \in C_2(G)$ with boundary a. Then, in section 1.6 we defined the filling norm in a following way:

$$\operatorname{fill}(a) = \lim_{n\to\infty} \frac{\|a^n\|_B}{n}.$$

The first thing we can do is to extend the filling norm to integral chains:

$$\operatorname{fill}\left(\sum_i g_i\right) = \lim_{n\to\infty} \frac{\|\sum_i g_i^n\|_B}{n}.$$

Then it can be extended by linearity to rational chains. Observe that for each fixed n we have

$$\left\|\sum_i g_i^n + \sum_j h_j^n\right\|_B \leqslant \left\|\sum_i g_i^n\right\|_B + \left\|\sum_j h_j^n\right\|_B ,$$

for any $g_i, h_j \in G$, and so fill is subadditive under addition of chains and can be extended by continuity to real chains. Summing up this discussion, we obtain the analogue of Lemma 1.6.5.

Lemma 1.9.4. For any finite linear chain $\sum_i t_i a_i \in B_1(G)$ there is an equality

$$\operatorname{scl}_G\left(\sum_i t_i a_i\right) = \frac{1}{4}\operatorname{fill}\left(\sum_i a_i\right).$$

Proof. Note that it is enough to prove the equality for integral chains.

Suppose that S is a surface with m boundary components, of genus $\operatorname{cl}_G(\sum_i a_i^n)$. We can construct a triangulation of S with $4\operatorname{cl}_G(\sum_i a_i^n) + 3m - 4$ triangles, with one vertex on each boundary component. Denote by T an embedded spanning tree in 1-skeleton, which connects the vertices on boundary components. Since there are m such vertices, T has $m-1$ edges.

Construct a simplicial 2-complex S/T as follows. Collapse T to a single point and then collapse all degenerate triangles. Initial triangulation of S induces a triangulation of S/T. Note that this simplicial complex has a single vertex, and its triangulation has fewer triangles than the triangulation of S, so S/T there is a group 2-chain A such that $\partial A = \sum_i c_i$, where each c_i is some conjugate to a_i, and

$$\|A\|_1 \leqslant 4\operatorname{cl}_G\left(\sum_i a_i^n\right) + 3m - 4.$$

Since the filling norm is constant on conjugacy classes, dividing by n and taking the limit as $n \to \infty$, we obtain

$$\operatorname{fill}\left(\sum_i a_i\right) \leqslant 4\operatorname{scl}_G\left(\sum_i a_i\right).$$

On the other hand, as before, we can rationally approximate the 2-chain A such that $\partial A = \sum_i a_i^n$ and the L^1-norm $\|A\|_1$ is close to $\|\sum_i a_i^n\|_B$, i.e. we may assume that A is rational. Furthermore, after scaling by some integer, we may assume that the chain A is integral. As in the proof of Lemma 1.6.5, we can group the edges of triangles in pairs to obtain a surface of the form S/T, which we constructed above. Now, by adding a cylindrical collar to each boundary components, which forces us to add $2m$ triangles, we can modify this surface to a genuine surface. Using Theorem A.4 and Proposition 1.9.3, we obtain

$$\operatorname{fill}\left(\sum_i a_i\right) \geqslant 4\,\mathrm{scl}_G\left(\sum_i t_i a_i\right),$$

and we are done. □

1.10 Generalized Bavard duality

Suppose G is a group and denote by $H(G)$ the subspace of $B_1(G)$ spanned by the elements of the form $g - hgh^{-1}$ and $g^n - ng$ for all $g, h \in G$ and all integers n. Denote

$$B_1^H(G) = B_1(G)/H(G).$$

Clearly, the function scl_G vanishes on all such elements, by construction, and therefore it vanishes on $H(G)$. We are now able to prove the generalized version of Theorem 1.6.6.

Theorem 1.10.1. Let G be a group. Then for any finite linear chain $\sum_i t_i a_i \in B_1^H(G)$ there is an equality

$$\mathrm{scl}_G\left(\sum_i t_i a_i\right) = \frac{1}{2} \sup_{\varphi \in Q_H(G)} \frac{\sum_i t_i \varphi(a_i)}{D(\varphi)},$$

where $Q_H(G) = Q(G)/H^1(G, \mathbb{R})$.

Proof. The proof of this theorem is precisely the same as the proof of Theorem 1.6.6 with Lemma 1.9.4 instead of Lemma 1.6.5.

□

The reason why it is better to restrict attention to the space $B_1^H(G)$ is that for some groups the function scl_G is a genuine norm on $B_1^H(G)$.

Theorem 1.10.2. Let $F = F\langle S\rangle$ be a free group. Then scl_F is a genuine norm on the vector space $B_1^H(F)$.

Proof. Consider a chain $c \in B_1^H(F)$. It has a representative of the form $\sum_i t_i \omega_i$, where each t_i is nonzero and each ω_i is a cyclically reduced primitive word in F and also for distinct i and j no two $\omega_i^{\pm 1}$ and $\omega_j^{\pm 1}$ are conjugate.

Note that we can reorder the elements of the chain c in such a way that the length of the word ω_1 is not less than the length of any ω_i. For an integer N consider the homogenization φ of the big counting quasimorphism $H_{\omega_1^N}$. We claim that for sufficiently large N there is an equality

$$\varphi(\omega_i) = 0$$

for all $i \neq 1$. Suppose that for some $i \neq 1$ the infinite product of words

$$\omega_i^\infty = \omega_i \omega_i \ldots$$

contains an arbitrarily big power ω_1^N as a subword, where $N > 0$. If $N = \frac{\text{lcm}(|\omega_1|,|\omega_i|)}{|\omega_1|}$, then $|\omega_1^N| = |\omega_i^M|$ and ω_1^N is conjugate to ω_i^M for some M:

$$\omega_1^N = t\omega_i^M t^{-1} = (t\omega_i t^{-1})^M.$$

Since ω_1 and $t\omega_i t^{-1}$ are primitive words, we have $M = N$, and therefore $\omega_1 = t\omega_i t^{-1}$, which leads to a contradiction. This proves the claim. Note that $\varphi(\omega_1) = \frac{1}{N}$. It follows that

$$\text{scl}_G(c) \geqslant \frac{|t_1|}{2ND(\varphi)} > 0,$$

and we are done. $\square$

Chapter 2

Free groups

In this chapter our goal is to prove that stable commutator length takes only rational values on the elements of a free group. First of all, we introduce the notion of branched surfaces as a toolbox to use in our proof of the rationality. Next, we prove that stable commutator length is rational for a certain class of elements in the free group of rank two, and then we extend our argument to arbitrary elements in a free group of arbitrary rank.

2.1 Branched surfaces

In this section we introduce the branched surfaces, following L. Mosher and U. Oertel [11].

Definition 2.1.1. A *branched surface* is a smooth, finite two-dimensional cell complex B such that for each $x \in B$, there is a unique tangent plane at x, and some neighbourhood of x is a union of smoothly embedded open disks all of which are tangent at x. In other words, a branched surface is a smooth, finite two-dimensional complex obtained from a finite collection of smooth surfaces by identifying compact subsurfaces. The *branch locus* of B, denoted $\mathrm{br}(B)$, is the set of points which are not 2-manifold points. The components of $B \setminus \mathrm{br}(B)$ are called the *sectors* of the branched surface. The set of sectors of B is denoted $S(B)$. A *simple branched surface* is a branched surface B with $\mathrm{br}(B)$ being a finite union of disjoint smoothly embedded simple loops and simple proper arcs.

Local sectors of a simple branched surface meet along segments of the branch locus. We consider branched surfaces with any number of local "sheets" approaching the branch segment from one of two sides. Figure 2.1 shows an example of a local model for a simple branched surface where five local sheets meet along the branch locus.

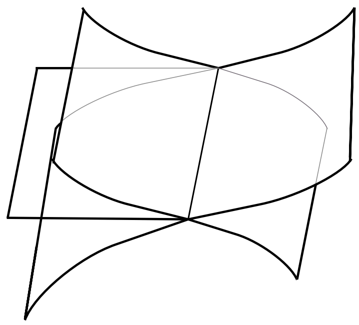

Figure 2.1: Local model for a simple branched surface.

It is to be noted that we consider branched surfaces with boundary and we also require that the branch locus intersects the boundary transversely. The sectors of a simple branched surface B are surfaces, possibly with boundary. A branched surface is oriented if the sectors can be compatibly oriented. In this discussion we only consider oriented branched surfaces. We now define a weight function on a simple branched surface.

Definition 2.1.2. Let B be a simple branched surface. A *weight* on B is a function $w : S(B) \longrightarrow \mathbb{R}$ such that for each component γ of the branch locus, the sum of the values of w on the sectors which meet γ on one side is equal to the sum of the values of w on the sectors which meet γ on the other side. We say that the weight w is *rational* if it takes only rational values and *integral* if it takes only integer values.

It is easy to see that the set of weights on B forms a subspace of $\mathbb{R}^{|S(B)|}$ defined by a finite family of linear inequalities. We denote the real vector space of weights on B by $W(B)$ and the convex cone of weights taking nonnegative values on each sector by $W^+(B)$.

Definition 2.1.3. Let B be an oriented simple branched surface. A *carrying map* is a proper orientation preserving immersion $f : S \longrightarrow B$, where S is some compact oriented surface S. We also say that B *carries* S.

Consider some carrying map $f : S \to B$. It is easy to see that f determines a nonnegative integral weight w_f with value on each sector $\sigma \in S(B)$ equal to the local degree of f along σ. Since f is an orientation preserving immersion, we have

$$w_f(\sigma) = \#\{f^{-1}(p) \mid p \in \sigma\}.$$

Lemma 2.1.4. Let B be a simple branched surface. Every nonnegative integral weight is represented by some carrying map. Furthermore, if $f : S \longrightarrow B$ represents a weight w_f, then $\chi(S)$ depends only on w_f and is a rational linear function of the coordinates of $w \in W(B)$.

Proof. Consider a nonnegative integral weight w. For each $\gamma \in \mathrm{br}(B)$ consider two sets X_1 and X_2, where X_1 contains $w(\sigma_1)$ copies of each sector $\sigma_1 \in S(B)$ approaching γ on one side and X_2 contains $w(\sigma_2)$ copies of each sector $\sigma_2 \in S(B)$ approaching γ on the other side. Since w is a weight, we have $|X_1| = |X_2|$, and so we can choose a one-to-one correspondence between X_1 and X_2. Now we can glue the pairs of copies according to this correspondence along the edges corresponding to γ. After this operation we end up with a surface S equipped with a tautological orientation preserving immersion to B, which determines the weight w.

We can consider each sector $\sigma \in S(B)$ as a surface with corners. The corners are the points where arcs of branch locus run into ∂B. Each such surface with corners has an *orbifold Euler characteristic* defined by

$$\chi_o(\sigma) = \chi(\sigma) - \frac{c(\sigma)}{4},$$

where $\chi(\sigma)$ is the Euler characteristic of the underlying surface and $c(\sigma)$ is the number of boundary corners of σ. If we obtain a surface S by gluing some finite number of surfaces S_i with corners, then

$$\chi(S) = \sum_i \chi_o(S_i).$$

It follows that if S is a surface with weight w, then

$$\chi(S) = \sum_\sigma w(\sigma)\chi_o(\sigma),$$

and we see that $\chi(S)$ depends only on w. □

On the other hand, the function χ^-, defined in section 1.3 might depend on the choice of a surface S, and, for instance, the number of disk components of S might depend on the way in which sectors are glued together.

Definition 2.1.5. An oriented simple branched surface B is said to be *essential* if it does not carry a disk or sphere.

For example, if every sector $\sigma \in S(B)$ satisfies $\chi_o(\sigma) \leqslant 0$, then we have $\chi(S) \leqslant 0$ for any surface S carried by B. It follows that in this case B is essential. If a surface S is carried by an essential simple branched surface, then for every component S_i of S we have $\chi(S_i) \leqslant 0$, and so $\chi^-(S) = \chi(S)$. It follows that if B is an essential simple branched surface and S is a surface with carrying map $f : S \longrightarrow B$ representing a nonnegative integral weight w_f, then $-\chi^-(S)$ is a linear function of w_f.

2.2 Alternating words

In this section we consider the free group on two generators: $F = F\langle a, b\rangle$, and prove the rationality of scl_F on the *alternating words* in F.

Definition 2.2.1. A word $\omega \in F$ is *alternating* if it has even length, and its letters alternate between one of $a^{\pm 1}$ and $b^{\pm 1}$.

Clearly, every alternating word is cyclically reduced. An alternating word ω is a representative of the derived subgroup $[F, F]$ if the number of letters a in ω is the same as the number of letters a^{-1} in ω, and similarly for b and b^{-1}. It follows that $|\omega| = 4k$ for some integer k. For example, the words $aba^{-1}b^{-1}$ and $a^{-1}bab^{-1}aba^{-1}b^{-1}$ are alternating.

Consider a handlebody H of genus two. For convenience, we think of H as the union of two handles H^+ and H^- glued along the *splitting* disk E. Denote by D^+ and D^- the *compressing* disks for the meridians of H^+ and H^- respectively (Figure 2.2).

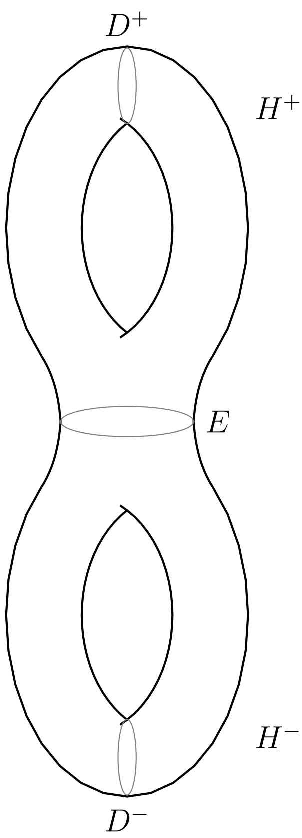

Figure 2.2: A handlebody of genus two.

Now we can identify the fundamental group $\pi_1(H)$ with F. Suppose a is represented by the core of the handle H^- and b is represented by the core of the handle H^+. An alternating word ω can be represented by a free homotopy class of loop in H as a union of arcs from E to itself, which run around either H^+ or H^- and intersect D^+ or D^- in a single point. We say that such loop or its homotopy class is in *bridge position* and we also assume that such loop is embedded in H.

Consider an alternating word ω and suppose that it is represented by a loop γ, which is in bridge position. Suppose ω has the form

$$\omega = a^{p_1}b^{q_1}a^{p_2}b^{q_2}...a^{p_m}b^{q_m},$$

where $p_i, q_i \in \{-1, 1\}$ for $1 \leqslant i \leqslant m$ and $m = \frac{|\omega|}{2}$ is even. Then γ can be represented as a union of arcs from E to itself:

$$\gamma = \alpha_1 \cup \beta_1 \cup ... \cup \alpha_m \cup \beta_m,$$

where each α_i is properly embedded in H^- and run in the direction determined by p_i, and each β_i is properly embedded in H^+ and run in the direction determined by q_i. Furthermore, each α_i and each β_i is oriented, and the initial point of each β_i is equal to the end point of the corresponding α_i. The initial point of each α_i is equal to the end point of β_{i-1} (the initial point of α_1 is the same as the end point of β_m).

Let S be a surface together with a map $f : (S, \partial S) \to (H, \gamma)$ satisfying

$$f_*[\partial S] = n(S)[\gamma].$$

By Proposition 1.3.1 we know that

$$\mathrm{scl}_F(\omega) = \inf_S \frac{-\chi^-(S)}{2n(S)}.$$

We are going to "simplify" the function f by obtaining a simpler surface S' from S with $n(S') = n(S)$ and $-\chi^-(S') < -\chi^-(S)$.

Note that we can assume that S has no disc components, no closed components, and no simple compressing loops, since otherwise we can reduce $-\chi^-(S)$ without changing $n(S)$. We can also reduce the value of $-\chi^-(S)$ by compressing those boundary components which map to γ with zero degree, so we can assume that every boundary component maps to γ with some nonzero degree.

We homotope f in such a way that its restriction to each boundary component in a covering map to γ and then perturb f rel. boundary to an immersion in general position with respect to D^+ and D^-. After this operation, $f^{-1}(D^+) \cap S$ is a union of disjoint, properly embedded arcs and loops in S. Since we assume that S has no simple compressing loops, all the loops in S are inessential, so by a homotopy of f we can push them off D^+. In $f^{-1}(D^+)$ there are no inessential arcs, since the restriction of f to each of the boundary components is a covering map for a loop and we may assume that $f^{-1}(D^+)$ is a union of disjoint essential properly embedded arcs in S. We can do the same modification to $f^{-1}(D^-)$ and after this, $f^{-1}(D^+ \cup D^-)$ is a union $\mathcal{U}$ of disjoint essential properly embedded arcs. Denote by $\mathcal{R}$ a union of open regular neighbourhoods in S of the components of $\mathcal{U}$. The components of $\mathcal{R}$ are called *rectangles*.

Denote $\mathcal{D} = D^+ \cup D^-$. The complement of tubular neighbourhood $N(\mathcal{D})$ in H retracts down to E. In fact, there is a deformation retraction of pairs

$$(H \setminus N(\mathcal{D})), \gamma \cap (H \setminus N(\mathcal{D})) \longrightarrow (E, \gamma \cap E).$$

This retraction can be extended to a map $r : H \longrightarrow H$ which restricts to a homotopy equivalence of pairs from $(N(\mathcal{D}), \gamma \cap N(\mathcal{D}))$ to $(H \setminus E, \gamma \cap (H \setminus E))$. After composing f with such a retraction, we get a new map $\widetilde{f}$, which is homotopic to f, such that $\mathcal{R} = \widetilde{f}^{-1}(H \setminus E)$. Let $Z \subset E$ be the union of the endpoints of the horizontal and vertical arcs in γ and note that Z is finite. Each component of $\partial S \setminus \mathcal{R}$ either maps by $\widetilde{f}$ to a point in Z, or to a horizontal arc in γ. In the first case, we can collapse the component to a point in S by a homotopy equivalence, and so we may assume that every arc in $\partial S \setminus \mathcal{R}$ is a horizontal arc.

Consider now the components of $S \setminus \mathcal{R}$. Let P be such a component and suppose it is not a disk. Then it contains an essential simple loop γ'. Since E is a disk, $\widetilde{f}$ maps γ' to a homotopically trivial loop in E and we can compress S along γ, mapping the compressing disks to E. Since we assumed that S contains no simple compressing loops, it turns out that P is topologically a disk. In fact, P has the structure of a polygon, whose edges are the arcs of boundary components of $\mathcal{R}$, and horizontal arcs. Denote by $\mathcal{P}$ the union of these polygons, and let P_i be a typical polygon.

For each polygon P_i denote by $s(P_i)$ the number of edges of P_i and by $h(P_i)$ the number of edges formed by horizontal arcs. Observe that two adjacent edges of P_i cannot both be horizontal, so $s(P_i) \geqslant 2$ and $h(P_i) \leqslant \frac{s(P_i)}{2}$. There are $2(s(P_i) - h(P_i))$ corners of each P_i, so $\chi_o(P_i) = 1 - \frac{(s(P_i)-h(P_i))}{2}$, and we have

$$-\chi^-(S) = -\chi(S) = \sum_i \frac{(s(P_i) - h(P_i) - 2)}{2}.$$

Consider a single polygon P_i and suppose there exists a point p in $\gamma \cap E$ and two different *boundary edges* e_1, e_2 of P_i such that they both map to p. Boundary compression along an embedded arc in P_i which joins e_1 to e_2, reduces the value of $-\chi^-(S)$ by one, and so after several such operations we may assume that every P_i has at most $|\omega|$ boundary edges mapping to distinct points of $\gamma \cap E$.

Now we can summarize what we have done so far. We started with a surface S and a map $f : (S, \partial S) \longrightarrow (H, \gamma)$ and then we applied homotopy and compression and obtained a new surface S' with a map $\widetilde{f}$, such that S' can be decomposed into rectangles and polygons. Rectangles map over the handles of H and run between two arcs of γ and polygons map to E. Each rectangle is determined by the pair of arcs of γ, up to homotopy, and each polygon is determined up to homotopy by a cyclically ordered list of distinct elements of $\gamma \cap E$, which are the images of boundary edges, and by the information whether each component of a polygon in the closure of a rectangle bounds a rectangle mapping to H^+ or to H^-. We have only finitely many combinatorial possibilities for each rectangle and for each polygon, and so we can build a surface S' from finitely many pieces. This suggests that the computation of $\mathrm{scl}_F(\omega)$ can be reduced to a finite integer linear programming problem.

Suppose that B is an oriented essential simple branched surface constructed as follows. The sectors of B are represented by the disjoint union of all possible polygons with boundary edges mapping to distinct points of $\gamma \cap E$ and all possible rectangles. Then we glue the rectangles to polygons in all possible orientation preserving ways with the only condition that each component of a polygon in a closure of rectangle which bounds a rectangle mapping to H^+ is only glued to a rectangle in H^+, and each component of a polygon in a closure of rectangle which bounds a rectangle mapping to H^- is only glued to a rectangle in H^-. Thus, we have a branched surface B and a map $\iota : B \longrightarrow H$ which takes ∂B to γ.

For each pair of distinct points in $\gamma \cap E$, there are two components of $\mathrm{br}(B)$ distinguished by the information whether these components bound rectangles in H^+ or in H^-, so $\mathrm{br}(B)$ is a 1-manifold, which ensures us that B is a simple branched surface. Moreover, each rectangle contributes 0 to χ_o and each polygon contributes some nonpositive value, so we conclude that B does not carry a disk or sphere, i.e. it is essential.

Now, since every surface S with a map $f : (S, \partial S) \longrightarrow (H, \gamma)$ can be appropriately modified to a surface S' which can be decomposed into rectangles and polygons, we conclude that each such map f can be modified without changing the value of $-\chi^-(S)$ to a map which is carried by B.

Let $w \in W^+(B)$ and let $f : S \longrightarrow B$ be carrying map with weight w. The map $\iota \circ f : S \longrightarrow H$ takes ∂S to γ. From the results of section 2.1 we know that $-\chi^-(S)$

is a rational linear function $-\widetilde{\chi}^-(w)$. There is a rational linear map $\partial : V \longrightarrow \mathbb{R}$ where V is a rational subspace of $W(B)$, defined in a following way. Given a positive integral weight w, let S be a surface carried by B associated to w. Then define $\partial(w) = n(S)$ and extend by linearity to V. The inverse $\partial^{-1}(1) \cap W^+(B)$ is a closed rational polyhedron. Furthermore, by construction we have

$$\mathrm{scl}_F(\omega) = \inf_{w \in \partial^{-1}(1) \cap W^+(B)} \frac{-\widetilde{\chi}^-(w)}{2}.$$

Since $-\widetilde{\chi}^-$ is a rational linear function which is nonnegative on the cone $W^+(B)$, this infimum is realized, and it follows that $\mathrm{scl}_F(\omega)$ is rational.

2.3 The Rationality Theorem

In this section we extend the argument used in the proof of rationality of stable commutator length for alternating words in the free group of rank two, to prove the rationality in free groups of arbitrary rank.

Theorem 2.3.1. [Rationality Theorem] Let F be a free group of arbitrary rank. Then for all $g \in [F, F]$,

$$\mathrm{scl}_F(g) \in \mathbb{Q}.$$

Proof. Let $F = F\langle S \rangle$ be a free group on generators $s_i \in S$. For each i denote by H_i a solid torus with a disk E_i in its boundary, and let H be constructed from H_i by identifying E_i with a single disk E. Denote by D_i the decomposing disk of H_i and let $\mathcal{D}$ be the union of all D_i.

Suppose $\omega \in F$ is a cyclically reduced word. Its conjugacy class determines a free homotopy class of loop in H. Let γ be a representative in this homotopy class such that its intersection with $\mathcal{D}$ and E is simple.

An arc whose interior is properly embedded in some $H_i \setminus E$ and with endpoints on the disk E is called a *vertical arc*. Note that each vertical arc intersects some D_i transversely in one point. A *horizontal arc* is an arc embedded in E.

For each appearance of s_i or s_i^{-1} in ω, γ will have one vertical arc in H_i. Since ω is cyclically reduced, γ will have one horizontal arc between two consecutive appearances of s_i or s_i^{-1}. Note that this uniquely determines the homotopy class of γ. We say that a representative γ in the free homotopy class corresponding to the conjugacy class of ω, constructed as above, is in the *bridge position*.

Consider a finite collection of words $\omega_1, ..., \omega_n$ which are cyclically reduced in their conjugacy classes and corresponding loops $\gamma_1, ..., \gamma_n$ in bridge position in H. Let Γ denote the union of these loops and let S be a surface without disks or closed components, or simple compression loops, with a map $f : (S, \partial S) \longrightarrow (H, \Gamma)$. As in the previous section, we may assume that $f^{-1}(\mathcal{D})$ is a union of disjoint essential properly embedded arcs, and $\mathcal{R} = f^{-1}(H \setminus E)$ is a union of disjoint embedded rectangles. By assumption that S has no simple compressing loops, we conclude that every component P_i of $S \setminus \mathcal{R}$ is a polygon.

The edges of a polygon P_i in the closure of components of $\mathcal{R}$ are called *branch edges*. There are also two types of *boundary edges*, namely those which map to a single endpoint of some vertical arc of some γ_i and those mapping to a horizontal edge.

As in the previous section, if some polygon P_i has two different boundary edges e_1, e_2 such that they both map to the same point or horizontal arc in $\Gamma \cap E$, then we can perform a boundary compression of S which joins e_1 to e_2 and reduces the value of $-\chi^-(S)$, so we may assume that two different boundary edges of the same polygon map to different points or horizontal arcs in $\Gamma \cap E$.

Denote by $b(P_i)$ and $c(P_i)$ the number of branch edges of a polygon P_i and the number of corners of P_i respectively. Then there are twice as many corners of P_i as branch edges, i.e. $c(P_i) = 2b(P_i)$, so we have $\chi_o(P_i) = 1 - \frac{b(P_i)}{2}$. Note that each rectangle contribute 0 to χ_o, so

$$-\chi^-(S) = -\chi(S) = \sum_i \frac{b(P_i) - 2}{2}.$$

As before, we can build an essential simple branched surface B, together with a map $\iota : B \longrightarrow H$ with $\iota(\partial B) = \Gamma$. Every map $f : (S, \partial S) \longrightarrow (H, \Gamma)$ can be appropriately modified so that the resulting map factors through a carrying map to B.

Let K be the kernel of the inclusion map $H_1(\Gamma, \mathbb{R}) \to H_1(H, \mathbb{R})$ and let K^+ be the intersection of K with the orthant spanned by all nonnegative combinations of the $[\gamma_i] \in H_1(\Gamma, \mathbb{R})$. With notation as in the previous section, there is a rational linear map $\partial : W^+(B) \longrightarrow K^+$, and for each $k \in K^+$ there is an equality

$$\mathrm{scl}_F(k) = \inf_{w \in \partial^{-1}(k) \cap W^+(B)} \frac{-\widetilde{\chi}^-(w)}{2}.$$

As in the previous section, the infimum is realized, and it follows that scl_F is rational in free groups. $\square$

Appendix A

Hyperbolic surfaces

We define a conformal structure on a surface S as an *atlas*. An *atlas* is a topological notion which is used to describe a surface. A *chart* for a surface S (equivalently, *coordinate chart* or *coordinate map*) is a homeomorphism φ from an open subset U of S to an open subset of the complex plane $\mathbb{C}$. The chart is denoted as (U, φ). Then an atlas for a surface S is a collection $\{(U_i, \varphi_i)\}_{i \in I}$ of charts on S such that

$$\bigcup_{i \in I} U_i = S.$$

Two atlases are said to be *compatible* if their union is an atlas satisfying the properties from the definition of an atlas. A maximal union of compatible atlases is then called a *maximal atlas*.

A *conformal structure* on a surface S is a maximal atlas $\{(U_i, \varphi_i)\}_{i \in I}$ such that for all $i, j \in I$, the map $\varphi_i \circ \varphi_j^{-1}$, which is called a *transition map*, and which is defined on $\varphi_j(U_i \cap U_j)$, is conformal, i.e. it preserves angles. Furthermore, we suppose that each puncture of S has a neighbourhood which is conformally equivalent to a punctured disk in $\mathbb{C}$.

An orientable surface with a conformal structure is also called a *Riemann surface*. Let S be an arbitrary triangulated surface. Then, by taking each triangle to be an equilateral Euclidean triangle with side length equal to one, and all gluing maps between edges to be isometries, we see that every surface can admit a conformal structure.

Any conformal structure defined on a surface S induces a conformal structure on $\overline{S}$. A conformal structure on S is said to be *conformally finite*, if it is conformally equivalent to a closed surface without finitely many points. Note that every surface of finite type admits a conformal structure which is conformally finite.

A Riemann surface with curvature -1 is called *hyperbolic*. A conformally finite surface S is hyperbolic if and only if $\chi(S) < 0$. From the Gauss-Bonnet Theorem, we have

$$\int_S K = 2\pi\chi(S),$$

where S is a closed Riemann surface and K is the Gaussian curvature. Thus, if S is hyperbolic, it follows that

$$\text{area}(S) = -2\pi\chi(S).$$

The Gauss-Bonnet Theorem also gives us the relationship between the area of a geodesic triangle Δ in $\mathbb{H}^2$ and its interior angles:

$$\text{area}(\Delta) = \pi - \alpha - \beta - \gamma,$$

where α, β, γ are the interior angles of Δ. Since the area is positive, it follows that $\text{area}(\Delta) \leqslant \pi$. Note that we allow some of the vertices of Δ to lie at infinity.

Definition A.1. Let M me a hyperbolic m-manifold, and let $\sigma : \Delta^n \longrightarrow M$ be a singular n-simplex. Define the *straightening* σ_g of σ as follows. First, lift σ to a map $\widetilde{\sigma} : \Delta^n \longrightarrow \mathbb{H}^m$. Denote the vertices of Σ^n by $v_0, ..., v_n$. In the hyperboloid model of hyperbolic geometry, $\mathbb{H}^m$ is the positive sheet (i.e. $x_{m+1} > 0$) of the hyperboloid $\|x\| = -1$ in $\mathbb{R}^{m+1}$ with the inner product

$$\|x\| = x_1^2 + x_2^2 + ... + x_m^2 - x_{m+1}^2.$$

If $t_0, ..., t_n$ represent the barycentric coordinates on Δ^n, so that $v = \sum_i t_i v_i$ is a point in Δ^n, define

$$\widetilde{\sigma}_g(v) = \frac{\sum_{i=1}^n t_i \widetilde{\sigma}(v_i)}{-\| \sum_{i=1}^n t_i \widetilde{\sigma}(v_i) \|},$$

and define σ_g to be the composition of $\widetilde{\sigma}$ with projection $\mathbb{H}^m \longrightarrow M$.

The isometry group of $\mathbb{H}^m$ acts linearly on $\mathbb{R}^{m+1}$, preserving $\| \cdot \|$, and so the straightening map $\sigma \mapsto \sigma_g$ is well-defined, and independent of the choice of lift.

For a hyperbolic manifold M, define

$$\text{str} : C_*(M) \longrightarrow C_*(M)$$

by $\text{str}(\sigma) = \sigma_g$, and extend by linearity.

Let S be a conformally finite surface, possibly with boundary. If S is closed and oriented, the fundamental class $[S]$ is the generator in $H_2(S, \partial S)$ which induces the orientation on S.

Definition A.2. The *Gromov norm* of S is defined as follows. Consider the homomorphism

$$i_* : H_2(S, \partial S, \mathbb{Z}) \longrightarrow H_2(S, \partial S, \mathbb{R}),$$

which is induced by inclusion $\mathbb{Z} \longrightarrow \mathbb{R}$, and let $C = \sum_i r_i \sigma_i$ represent the image of the fundamental class $[S]$ with $r_i \in \mathbb{R}$. Denote

$$\|C\|_1 = \sum_i |r_i|,$$

and set

$$\|[S]\|_1 = \inf_C \|C\|_1.$$

Lemma A.3. Let S be an orientable surface with p boundary components, where $p > 1$. Then for any integer $m > 1$ such that $\gcd(p-1, m) = 1$, there is an m-fold cyclic cover S_m with p boundary components, each of which maps to the corresponding component of ∂S by a m-fold covering.

Proof. Let S be such a surface. Then the inclusion map $\partial S \longrightarrow S$ induces a homomorphism $H_1(\partial S) \longrightarrow H_1(S)$ with one-dimensional kernel, generated by the homology class represented by the union ∂S. In particular, we can take $p-1$ boundary components to be the part of a basis for $H_1(S)$. Denote the images of the boundary components in $H_1(S)$ by $b_1, ..., b_p$ and take $b_1, ..., b_{p-1}$ to be the part of a basis for $H_1(S)$. If $\gcd(p-1, m) = 1$, let $\alpha \in H^1(S, \mathbb{Z}/m\mathbb{Z}) = \mathrm{Hom}(H_1(S), \mathbb{Z}/m\mathbb{Z})$ be such that $\alpha(b_i) = 1$ for $1 \leqslant i \leqslant p-1$. Then for all $1 \leqslant j \leqslant p$, $\alpha(b_j)$ is primitive, with kernel defining a regular m-fold cover S_m with p boundary components. □

Theorem A.4. Let S be a compact orientable surface with $\chi(S) < 0$. Then

$$\|[S]\|_1 = -2\chi(S).$$

Proof. Let S be a surface of genus g with p boundary components, so

$$\chi(S) = 2 - 2g - p.$$

There is a triangulation of S with one vertex on each boundary component, and with a total of $4g + 3p - 4$ triangles. By the previous lemma, there is an m-fold cover S_m with p boundary components. Euler characteristic is multiplicative under taking covers, so $\chi(S_m) = m\chi(S) = 2m - 2gm - mp$, and S_m admits a triangulation with $(4g + 2p - 4)m + p$ triangles. Taking the projection of this triangulation under the covering map $S_m \longrightarrow S$ gives us a representative C of $m[S]$ with L^1-norm equal to $(4g+2p-4)m+p$. Dividing by m and taking a limit as $m \to \infty$, we obtain

$$\|[S]\|_1 \leqslant -2\chi(S).$$

Conversely, let $C = \sum_i r_i \sigma_i$ be any representative of the fundamental class $[S]$. Then $\|C\|_1 \geqslant \|\operatorname{str}(C)\|_1$, and since the area of any geodesic triangle is no more than π, and

$$\sum_i |r_i|\pi \geqslant \operatorname{area}(S) = -2\pi\chi(S),$$

we get

$$\|[S]\|_1 \geqslant \|\operatorname{str}(C)\|_1 = \sum_i |r_i| \geqslant -2\chi(S).$$

The proof follows. □

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